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LOCALLY DEFINED FUNCTIONS AND THEIR APPROACH BY A POLYNOMIAL

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ABSTRACT

It is possible to obtain a polynomial that approximates a given function, in principle without the need for the function to be derivable in any number. Given a locally defined function it can be expressed as the sum of a polynomial of given degree plus the product of a power of the same degree by a locally defined function that only depends on such function and of a given number in its domain of it. The derivation coefficients are calculated iteratively and in this case the higher order derivation quotients are named. If the quotient of a certain order is bounded, the polynomial approximates the given function and its degree equals the order previous to that bounded derivation quotient.

KEYWORDS– LocallyDefined, locally bounded, Taylor's Theorem, Derivation Quotient, High order Derivation Qoutient.

INTRODUCTION

The difference of a given function with its Mc Laurin polynomial, called the remainder of such function, is the product of the degree power equal to such polynomial by a function that tends to zero when the independent variable tends to zero in the case that the given function is differentiable a number of times equal to the degree of the Mac Laurin polynomial: the special case of Taylor's Theorem.

In the complex variable function theory given in [1], the Taylor polynomial has been obtained in an iterative form, more precisely as a succession of derivation ratios for complex functions. Since such a procedure only depends on the algebra that is common to the numbers of the line and the plane, we have been able to carry it out in the present case, that is, for functions with values in the interval of all the numbers (number line, i.e., real numbers) and defined locally.

Methodology

Given a locally defined function, we define the derivation quotient for the given function, called the first order derivation quotient, then we can define the derivation quotient of the second order, thirdorder derivation quotient and so on until any order derivation quotient, i.e., it can construct the derivation quotient of any order.

Objectives

Give a direct demonstration of Taylor's Theorem so that it can be easily reproduced by students, instructors,researchers, etc., for functions not necessarily differentiable with a residual of the given function thatis defined locally at the point of development of the Taylor polynomial andis notnecessarily be continuous in some number [2].

DEFINITIONS AND SOME RESULTS ABOUT LOCALLY DEFINED FUNCTIONS

Definitions:

1. A number in the domain of a function is said to be a pre-image or simply a number of the given function.
2. A number in the range of a function is called the value of the given.
3. It is said that $x \neq a$ is close to a if and only if $|x - a| < \delta$ for some $\delta = \delta(a)$.
4. A function is said to be locally defined in a given number a if it is defined for all pre-images (numbers of the function) close to the number a.

Example: The function $x \mapsto sgn(x) = \frac{x}{|x|}$ is locally defined at 0, since it is defined for $x \neq 0$ and close to 0.

Remark: The functions considered here do not need to be defined globally.

THE QUOTIENTS OF DERIVATION

The development of a locally bounded function is performed from the iterated definition of the

derivation quotients of a given order.

Definition: Given the function$f$locally defined in a and a number$l$, defines the derivation quotient in$a$of f by$F_l(x) = \frac{f(x)-l}{x-a}$Eq. 1 *if and only if* $f(x) = f(a) + (x-a)F_1(x-a)$Eq. 2.

Proposition. $f$is locally defined in a if and only if$F_l(x) = \frac{f(x)-l}{x-a}$Eq. 3is locally defined in$a$.

Remark.f is defined on$D - \{ a \}$ if and only if $F_l$is defined in the same set$D - \{ a \}$

Proof. It is immediate in view of above definition. *since* $x \neq a$.*Given the numbers* $l_1, , \ldots l_n$, *the function* f *is locally defined in a if and only if the higher order derivation ratios by iteration* $F_m(x) \frac{F_{m-1}(x)-l_{m-1}}{x-a}, 1 \leq m \leq n$, *Eq. 4namely*

$F_1(x) = \frac{f(x)-l_1}{x-a}, F_2(x) = \frac{F_1(x)-l_2}{x-a}, \ldots, F_n(x) = \frac{F_{n-1}(x)-l_{n-1}}{x-a}$, *Eq.* 5if and only if

$$\begin{aligned} f(x) &= l + (x-a)F_1(x) \\ F_1(x) &= l_1 + (x-a)F_2(x) \\ &\vdots \\ F_{n-1}(x) &= l_{n-1} + (x-a)F_n(x) \end{aligned}$$

*Eq. 6*are locally defined in$a$

Proof.By Proposition *1*, $F_m = \frac{F_{m-1}(x)-l_{m.1}}{x-a}$*Eq. 7 for* $1 \leq m \leq n$if and only if $F_{m-1}(x) = l_{m-1} + (x-a)F_m(x)$,*Eq. 8*if and only if f locally is locally defined in a or explicitly

$$\begin{aligned} f(x) &= l + (x-a)F_1(x) \\ F_1(x) &= l_1 + (x-a)F_2(x) \\ &\vdots \\ F_{n-1}(x) &= l_{n-1} + (x-a)F_n(x) \end{aligned}$$

Eq. 9.

Theorem.f is defined locally in a, one has that

$f(x) = l + l_1(x-a) + l_2(x-a)^2 + l_3(x-a)^3 + \cdots + l_{n-1}(x-a)^{n-1} + (x - anFnx$Eq. 10

Proof. ReplacingF_1in the first relation defined for $f(x)$, it is obtained that

$f(x) = l + l_1(x-a) + l_2(x-a)^2F_2(x)$, substituting $F_2(x)$in the result, we have

$= l + l_1(x-a) + l_2(x-a)^2 + (x-a)^3F_3(x)$,andsoon has been replaced$F_n$to obtain

$= l + l_1(x-a) + l_2(x-a)^2 + l_3(x-a)^3 + \cdots + l_{n-1}(x-a)^{n-1} + (x-a)^nF_n(x)$. Eq. 11

Corollary. Whenever$F_n$is locally bound, it hasthat$f(x) = l + l_1(x-a) + l_2(x-a)^2 + l_3(x-a)^3 + \cdots + l_{n-1}(x-a)^{n-1} + (x-a)^{n-1}h_{n-1}(x)$, Eq. 12, where $\lim_{x \to a} h_{n-1}(x) = 0$

Proof. By the above Theorem, $f(x) = l + l_1(x-a) + l_2(x-a)^2 + l_3(x-a)^3 + \cdots + l_{n-1}(x-a)^{n-1} + (x-a)^{n-1}(F_{n-1}(x) - l_{n-1})$.*Eq.* *13*Placing$h_{n-1}(x) = F_{n-1}(x) - l_{n-1}$, we have that $\lim_{x\mapsto a} h_n(x) = 0$, *Eq. 14 since* $\left|\frac{F_{n-1}(x)-l_{n-1}}{x-a}\right| = |F_n(x)| \leq M_a$ *Eq. 15* locally in a due to the hypothesis that F_n is locally bounded in a.

Example: The function $x \to \frac{1}{1-x}$ is locally defined at 0. In this case

$F_1(x) = \frac{\frac{1}{1-x}-\frac{1}{1-0}}{x-0} = \frac{\frac{1-(11-x)}{1-x}}{x-0} = \frac{\frac{x}{1-x}}{x} = \frac{1}{1-x}$, so $F_n(x) = \frac{1}{1-x}$ and therefore F_n is locally defined at 0 and continuous there so that is locally bounded as well. In this way there exists the polynomial of derivation coefficients of any order, namely

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n\left(\frac{1}{1-x} - \frac{1}{1-0}\right)$$

$$= 1 + x + x^2 + \cdots + x^n + x^n\,\frac{1-(1-x)}{1-x}$$

$= 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}$, which is Taylor's Theorem for the assumed function

CONCLUSIONS

The polynomial coefficients presented here are calculated with the higher order derivationcoefficients (secants) in a given number, so in principle it is not necessary to calculate the higher order derivatives for calculing the coeficieents of such polynomial. This polynomial is an approximation in case that the derivationquotient of certain order is locally bounded, that is, for pre-images close to the number where the development in a polinomial of the given function is performed. In some cases as rhe function in the example given above, the polynomial calculated with the derivation quotients agrees with the Taylor's polynomial. Therefore, it may be not necessary calculing the higher orden derivatives.The present aproach allows the calculation of the logarithm and the arctangent functions since they are related by means of the integral. In consequence, the present calculation could be useful in the Calculus classroom, not only of the differential one but also for integral one.

REFERENCES

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