Application of Fourier-Mellin Transform to Partial Differential equation

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Abstract

Partial differential equations have big importance in Mathematics and other fields of science. Therefore it is very important to know methods to solve such partial differential equations. One of the most known methods to solve partial differential equations is the integral transform method. Integral transforms, especially those named for Fourier and Mellin are well known as providing techniques for solving problems in linear systems. Characteristically, one uses the transformation as a mathematical or physical tool to alter the problem into one that can be solved.

In this paper, we have defined differentiation property of Fourier-Mellin Transform and using this property we have solve some partial differential equation such as wave equation, Laplace equation and Heat flow equation.

Keywords

Partial Differential equations, Fourier transform (FT), Mellin Transform (MT), Fourier-Mellin Transform (FMT).

1. Introduction

Mathematics is everywhere in every phenomenon, technology, observation, experiment etc. All we need to do is to understand the logic hidden behind. Since mathematical calculations give way to the ultimate results of every experiment. Partial differential equations have big importance in Mathematics and other fields of science. Therefore it is very important to know methods to solve such partial differential equations. One of the most known methods to solve partial differential equations is the integral transform method [2]. From the time of Laplace up to the present time, different theories of integral transforms have been proposed by the help of integrals with different kernels and ranges of integrations, chosen suitably. These integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. The most useful significance of integral transforms lies in the fact that they transform a class of differential equations into a class of algebraic equations, so that solutions of those differential equations can be obtained easily by algebraic methods and by use of results of integral transforms [5]. Linear integral transforms, especially those named for Fourier and Mellin are well known as providing techniques for solving problems in linear systems. Characteristically, one uses the transformation as a mathematical or physical tool to alter the problem into one that can be solved.

Communication is all based on mathematics, be it digital, wired or wireless. So for understanding the communication technology, the processes of modulation, demodulation and Fourier Transform need to be explored first. Humans, very easily perform FT mechanically almost every day without having idea of it. Fourier Transform can be used to convert from the series of numbers to sound [4].Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CD's and DVD's, cell phones, movies, computer graphics, image processing, and fingerprint analysis and storage, are, in one way or another, founded upon the many ramifications of Fourier theory [4].

The use of Mellin Integral transform is to derive different properties in statistics and probability densities of single continuous random variable [3]. The Mellin Transform is widely used in Computer Science because of its scale invariance property [7].

These Fourier and Mellin transforms have various uses in many fields separately. However there is much scope in extending double transformation to a certain class of generalized functions [1]. On combining these two transforms i.e. Fourier and Mellin transforms also used for solving differential and integral equations [6]. We have defined Fourier-Mellin Transform in integral form using as-

1.1. Definition of Fourier-Mellin Transform

The Fourier-Mellin Transform with parameter of $f(t,x)$ denoted by

$$FM\{f(t,x)\}(s,p) = F(s,p) = \int_0^\infty \int_0^\infty f(t,l,x,y)\ e^{-ist} x^{p-1}\ dt\ dx \qquad (1.1.1)$$

where, $K(t,x,s,p) = e^{-ist} x^{p-1}$ and $t(0 < t < \infty), x(0 < x < \infty)$.

In the present paper, we have defined the differentiation property of Fourier-Mellin Transform in section 2. Then, we have applied Fourier-Mellin Transform to Wave equation, Laplace equation and one dimensional Heat flow equation and solved one example of each equation using initial and boundary conditions.

2. Differentiation Property

2.1. Result

$FM\{f_t(t,x)\}(s,p) = is\ FM\{f(t,x)\}(s,p) - k$

Proof:

$FM\{f_t(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f_t(t,x)\ e^{-ist} x^{p-1}\ dt\ dx$

$FM\{f_t(t,x)\}(s,p) = is\ FM\{f(t,x)\}(s,p) - k$, where $\int_0^\infty f(0,x) x^{p-1}\ dx = k$ (2.1.1)

2.2. Result

$FM\{f_{tt}(t,x)\}(s,p) = (is)^2\ FM\{f(t,x)\}(s,p) - isk$ (2.2.1)

Similarly, $FM\{f_{ttt}(t,x)\}(s,p) = (is)^3 FM\{f(t,x)\} - (is)^2 k$ (2.2.2)

and $FM\{f_n(t,x)\}(s,p) = (is)^n FM\{f(t,x)\} - (is)^{n-1} k$ (2.2.3)

3. Wave Equation

3.1. One dimensional wave equation is solved by using Fourier- Mellin integral Transform.

The wave equation is $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$ i.e. $f_{tt}(t,x) = c^2 f_{xx}(t,x)$, where $c^2 = \frac{1}{\rho s}$.

Proof: The Fourier-Mellin transformation is

$FM\{f(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f_t(t,x)\ e^{-ist} x^{p-1}\ dt\ dx$

Now we applying the Fourier-Mellin Transform to

$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$ i.e. $f_{tt}(t,x) = c^2 f_{xx}(t,x)$

Therefore, $(is)^2 FM\{f(t,x)\} - isk = c^2 D_x^2\ FM\{f(t,x)\}$

Therefore, we get $\left(D_x^2 + \frac{s^2}{c^2}\right)\ FM\{f(t,x)\} = -\frac{isk}{c^2}$ (3.1.1)

This is ordinary differential equation w.r.t. x. Now, its auxiliary equation is

Therefore, $C.F. = c_1 \cos\frac{s}{c}x + c_2 \sin\frac{s}{c}x$ (3.1.2)

Also, we have $P.I. = -\frac{ik}{s}$ (3.1.3)

Therefore, its complete solution is-

$$FM\{f(t,x)\} = c_1 \cos\frac{s}{c}x + c_2 \sin\frac{s}{c}x - \frac{ik}{s} \qquad (3.1.4)$$

3.2. Examples

To illustrate the use of the Fourier-Mellin Transform in solving the certain partial differential equation. We Propose to find the solution $f(t,x)$ of the equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$, satisfying the boundary conditions, The initial and boundary condition are i) If $x = 0$ then $f(t,0) = 0$ ii) If $x = a$ then $f(t,a) = 0$.

Solution: The Solution of wave equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$ is

$$FM\{f(t,x)\} = c_1 \cos\frac{s}{c}x + c_2 \sin\frac{s}{c}x - \frac{ik}{s} \qquad (3.2.1)$$

i) If $x = 0$ then $f(t,0) = 0$ then (3.2.1) is

$FM\{f(t,0)\} = c_1 \cos 0 + c_2 \sin 0 - \frac{ik}{s}$. Therefore, $c_1 = \frac{ik}{s}$

ii) If $x = a$ then $f(t,a) = 0$

$FM\{f(t,a)\} = c_1 \cos\frac{s}{c}a + c_2 \sin\frac{s}{c}a - \frac{ik}{s}$. So, $c_2 = \frac{ik}{s}\left[\operatorname{cosec}\frac{s}{c}a - \cot\frac{s}{c}a\right]$

Using the values of c_1 and c_2 in (2.2.1) we have

$$FM\{f(t,x)\} = \frac{ik}{s}\left\{\cos\frac{s}{c}x + \left[\operatorname{cosec}\frac{s}{c}a - \cot\frac{s}{c}a\right]\sin\frac{s}{c}x - 1\right\}$$

4. Laplace Equation

4.1. Laplace equation in Cartesian form is solved by using Fourier-Mellin Transform.

The Laplace equation in Cartesian form is $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$ that is $f_{tt}(t,x) + f_{xx}(t,x) = 0$.

Proof: The Fourier-Mellin Transform is

$$FM\{f(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f_t(t,x)\ e^{-ist} x^{p-1}\ dt\ dx \qquad (4.1.1)$$

Now applying Fourier-Mellin Transform to Laplace equation $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$

We have

$FM\{f_{tt}(t,x)\} + FM\{f_{xx}(t,x)\} = 0$

$(is)^2 FM\{f(t,x)\} - isk + D_x^2\ FM\{f(t,x)\} = 0$

$(D_x^2 - s^2)FM\{f(t,x)\} = isk$ (4.1.2)

This is ordinary differential equation w.r.t. y.

$C.F. = c_1 e^{sx} + c_2 e^{-sx}$ (4.1.3)

Also, $P.I. = \frac{-ik}{s}$ (4.1.4)

Its complete solution is

$FM\{f(t,x)\} = c_1 e^{sx} + c_2 e^{-sx} - \frac{ik}{s}$ (4.1.5)

4.2. Example

To illustrate the use of the Fourier-Mellin Transform in solving certain partial differential equation. We proposed to find the solution $f(t,x)$ of $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$, satisfying the boundary conditions:- i) If $x = 0$ then $f(t,0) = 0$ ii) If $x = a$ then $f(t,a) = 0$.

Solution: The Solution of the partial differential equation $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is given by

$FM\{f(t,x)\} = c_1 e^{sx} + c_2 e^{-sx} - \frac{ik}{s}$ (4.2.1)

i) If $x = 0$ then $f(t,0) = 0$

Then (4.2.1) become $FM\{f(t,0)\} = c_1 + c_2 - \frac{ik}{s}$

$c_1 + c_2 = \frac{ik_1}{s}$ (4.2.2)

ii) If $x = a$ then $f(t,a) = 0$. Then (4.2.2) become-

$FM\{f(t,a)\} = c_1 e^{sa} + c_2 e^{-sa} - \frac{ik}{s}$

$c_1 e^{sa} + c_2 e^{-sa} = \frac{ik_1}{s}$ (4.2.3)

By solving equation (4.2.2) and (4.2.3), we get

$c_2 = \frac{ik}{s}\frac{(e^{sa}-1)}{(e^{sa}-e^{-sa})}$ and $c_1 = -\frac{ik}{s}\frac{(e^{-sa}-1)}{(e^{sa}-e^{-sa})}$

Putting above values in equation in (4.2.1) we get-

$$FM\{f(t,x)\} = \frac{ik}{2s \sinh\ sa}[(e^{sa}-1)e^{sy} - (e^{-sa}-1)e^{-sy} - 2\sinh sa]$$

5. Heat Flow equation

5.1. Heat Flow equation is solved by using Fourier-Mellin integral Transform.

The equation of heat flow is $\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2}$, where $c^2 = \frac{1}{\rho s}$.

Proof: The Fourier-Mellin integral transformation is

$$FM\{f(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f_t(t,x)\ e^{-ist} x^{p-1}\ dt\ dx$$

Then, we have

$$FM\{f_x(t,x)\} = c^2 FM\{f_{xx}(t,x)\}$$

$$\left(D_x^2 - \frac{is}{c^2}\right) FM\{f(t,x)\} = -\frac{k}{c^2} \qquad (5.1.1)$$

This is ordinary differential equation w.r.t. x. So, we have its auxiliary equation-

$$C.F. = c_1 \cos\frac{\sqrt{s}}{\sqrt{i}\,c} x + c_2\ \sin\frac{\sqrt{s}}{\sqrt{i}\,c} x \qquad (5.1.2)$$

Also, $P.I. = \frac{k}{is}$ (5.1.3)

Therefore, the complete solution is

$$FM\{f(t,x)\} = c_1 \cos\frac{\sqrt{s}}{\sqrt{i}\,c} x + c_2\ \sin\frac{\sqrt{s}}{\sqrt{i}\,c} x + \frac{k_1}{is} \qquad (5.1.4)$$

5.2. Example

To illustrate the use of Fourier-Mellin integral transform in solving the certain partial differential equation $\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2}$. We proposed to find the solution satisfying the boundary conditions

i) If $x = 0$ then $f(t,0) = 0$ ii) If $x = a$ then $f(t,a) = 0$.

Solution: We have Heat flow equation as $\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2}$,

Its solution is as given $FM\{f(t,x)\} = c_1 \cos\frac{\sqrt{s}}{\sqrt{i}\,c}x + c_2 \sin\frac{\sqrt{s}}{\sqrt{i}\,c}x + \frac{k}{is}$ (5.2.1)

Now, we have to find the solution of above heat flow equation by using the boundary conditions as i) If $x = 0$ then $f(t,0) = 0$. Then (5.2.1) becomes $c_1 = -\frac{k}{is}$ (5.2.2)

ii) If $x = a$ then $f(t,a) = 0$. Then (5.2.1) becomes $c_2 = \frac{k}{is}\left[\cot\frac{\sqrt{s}}{\sqrt{ic}}a - \operatorname{cosec}\frac{\sqrt{s}}{\sqrt{ic}}a\right]$ (5.2.3)

$$\therefore FM\{f(t,x)\} = \frac{k}{is}\left\{\left[\cot\frac{\sqrt{s}}{\sqrt{ic}}a - \operatorname{cosec}\frac{\sqrt{s}}{\sqrt{ic}}a\right]\sin\frac{\sqrt{s}}{\sqrt{ic}}x - \cos\frac{\sqrt{s}}{\sqrt{ic}}x + 1\right\}$$

Conclusion

In this paper we have solved the wave equation, Laplace equation and Heat flow equation with one example of each equation using the differentiation property of Fourier-Mellin Transform.

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