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STUDY ON FUZZY G-PRIMARY IDEALS

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ABSTRACT: After the fuzzy logic which is presented by its founder"Zadeh",one of the important topics in mathematics is the application of fuzzy logic in algebraic structures, on this way many articles and speeches were published by mathematicians in the field of fuzzy algebra. In this paper, it is presented some definitions of fuzzy G-algebraic structures in commutative algebra, fuzzy G-subdomains and fuzzy G-Ideals are expressed and have been proved some theorems and related corollaries.

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1. Introduction

The main object in this paper is introducing some symbols in the field of fuzzy *G*-structures, fuzzy *G*- subdomains and fuzzy *G*-ideals, also it will be discussed some concepts and results of the fuzzy *G*- subdomains and fuzzy G-ideals, some definitions and sentences are obtained from the book of "Fuzzy Commutative Algebra" Mordeson[12] and some articles about the fuzzy rings and fuzzy ideals by various authors, Mashinchi and Zahedi]11[, Swami [15], Alizadeh [1,2]. About the concept of G- structures have been used the "Commutative Rings " Kaplansky [8] and other articles of Alizadeh [3]. However, the first part is expressed some of the basic concepts of fuzzy *G*-structures and the second phase is explained *G*- subdomains, the new definition also refers to the cases and the results have been remarkable, at the end part were discussed some properties of fuzzy *G*-ideals , the conclusion of this article have been raised the proving of Theorems and results.
In the following some operations on fuzzy structures in a commutative ring *R* are introduced. In addition the set of all fuzzy subsets on *R* denoted by *F(R).*

Definition 1.1. [3] Let R is a ring and $A \in F(R), A$ is called a fuzzy subring of R if for all $a, b \in R$:
$i)\ A(a-b) \geq \inf\{A(a), A(b)\} = (A(a) \wedge A(b))$.
$ii)\ A(ab) \geq inf\{A(a), A(b)\} = (A(a) \wedge A(b))$.

Definition 1.2 .*A* is a fuzzy subdomain of *R[x]* if:
$i)\ A\ is\ a\ fuzzy\ subring\ of\ R[x]$.
$ii)\ For\ all\ f, g \in R[x] if\ A(fg) = 0, then: A(f) = 0 or A(g) = 0$.
It is obvious that $A(fg) \geq inf\ \{A(f), A(g)\}$,because the *A* is a fuzzy subring of *R[x]*.

The set of all fuzzy subdomains of *R[x]* is denoted by $F_D(R[x])$.

Definition 1.3. For each $A \in F_D(R[x])$,the $A_t = \{f \in R[x] | A(f) \geq t\}, \forall t \in [0,1]$ is called *"t-cut"* or *"t- level set"*of *R[x]*.

Theorem 1.4. [1[$A \in F_D(R[x])$ *if and only if* $\forall t \in [0,1]$,*the t-level set of* A_t *is a subdomain of* $R[x]$.

Definition 1.5. A non-empty fuzzy subset A of *R[x]* is said to be an ideal if and only if, for any $f, g \in R[x]$:
$i)$ $A(f-g) \geq A(f) \wedge A(g)$
$ii)$ $A(fg) \geq A(f) \vee A(g)$.

Note 1.6.1) If R is a commutative ring then the condition of*"ii"*in definition of *"1.5"*is equivalent by following:
$A(fg) \geq A(f), \forall f, g \in R[x]$
2) The set of all fuzzy ideals on *R[x]* is denoted by $F_I(R[x])$.

Theorem 1.7.*Let* $A \in F_D(R[x])$, *then* $A \in F_I(R[x])$ *if and only if* A_t *is an ideal of* $R[x]$, $\forall t \in \{A(R[x])\}$.
Where, $A(R[x]) = \{A(f): f \in R[x]\}$.

Proof. Let $A \in F_I(R[x])$ and $t \in [0,1]$ be such that:

$$t \in \{A(R[x])\}$$

If $f, g \in A_t$ and $h \in R[x]$ then:

$$A(f-g) \geq \inf\{A(f), A(g)\} \geq t$$

And

$$A(hf) \geq A(f) \geq t$$

Hence $f-g, hf \in A_t$. Thus A_t is an ideal of *R[x]*.
Conversely, let A_t be an ideal of $R[x]$ and

$$\forall t \in \{A(R[x])\}$$

Let $f, g, h \in R[x]$ and $t = inf\{A(f), A(g)\}$,Then $f, g \in A_t$.
Thus: $f - g \in A_t$.
Hence:

$$\left(A(f-g)\right) \geq t = inf\{A(f), A(g)\}.$$

Let $s = A(f), A(f) \in A(R[x])$. Then $f \in A_s$. Thus $hf \in A_s$ since A_s is an ideal of *R[x]*. Hence $\left(A(hf)\right) \geq s = A(f)$.Thus A is a fuzzy ideal of *R[x]*. □

Remark 1.8. Let *D* be an integral domain with quotient field *K*, the following two statements are equivalent:
i) *K*is a finitely generated ring over *D*.
ii) *K* as a ring, can be generated over *D* by one element.

Definition 1.9 An integral domain satisfying either (hence both) of the statements in Remark*"1.8"*is called an *G-domain*.
\end{defn}
The name honors Oscar Goldman. His paper [7] appeared at virtually the same time as a similar paper by Krull [9]. Since Krull already has a class of rings named after him, it seems advisible not to attempt to honor Krull in this connection. Further results concerning the material in this section appear in Gilmer's paper [6].

Definition 1.10. [8] Let R be an integral domain and K be its quotient field, the R is called a *G-domain* if K as a ring on $R$is finitely generated and therefore can be generated by one element, this terminology honors the approach of Goldman to be used.

Note 1.11. [8] *1)* Let R be a G-domain with its quotient field K, let \$T\$ be a ring between R and K,then T is also a G-domain.
2) If R is an integral domain and x be an interment over R, then $R[x]$ is never a *G-domain*.
3) Let $R \subseteq T$ are integral domains and suppose that T is algebraic over R and finitely generated as a ring over R, then R is a G-domain if and only if T is a G-domain.

Remark 1.12. [14] Let $\Phi: R \longrightarrow S^{-1}R$ is the canonical ring homomorphism, in which S is a multiplicative closed subset of R, then:
1) If Q is a primary ideal of R, such that $Q \cap S \neq \emptyset\ , then\colon Q^e = S^{-1}R.$
2) If Q is a P-primary ideal of R and $Q \cap S = \emptyset$, then Q^e is a P^e-primary ideal of $S^{-1}R$.
3) If Q is an P-primary ideal of $S^{-1}R$, then Q^c is a P^c-primary ideal of R and $Q^c \cap S = \emptyset$, in addition $Q^{ce} = Q$.
4) All primary ideals of $S^{-1}R$ are exactly as the form of Q^e, where $Q$is a primary ideal of R and disjoint of S. In fact each primary ideal of $S^{-1}R$, for some primary ideal Q of R, such that $Q \cap S = \emptyset$ is exactly as the form of $Q^\wedge e$.

Definition 1.13. [3]A prime ideal P in an integral domain of R is called *G-ideal*, if *R/P* is a *G-domain*.

It is obvious that every maximal ideal in a domain is also a *G-ideal*.

2. Fuzzy G-Subdomains

Definition 2.1. Let A be a fuzzy subset of domain D, the D_A is a subdomain of D if it is generated by the set of $S = \{x \in D: 0 < A(x) \leq 1\}. (i.e., D_A$ is the intersection of all subdomains of D such that each of them contains the set of S). It is obvious that the D_A is the smallest subdomain of D in which contains S.

Lemma 2.2.*Let D is a domain, the {fuzzy} subset A is a {fuzzy} G-subdomain of D if D_A as a subdomain of D itself is a G-domain.*

Example 2.3. Let $\mathbb{Q}$ be the Rational numbers, since for each prime number *2, 3, ...* the extended fields $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{2},\sqrt{3}], \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}]\ and\ \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7}]$ are *G-domains*. If we define *A(x)* as the following:

$$A(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 1/2\,, x \in \mathbb{Q}[\sqrt{2}] - \mathbb{Q} \\ 1/3\ , & x \in \mathbb{Q}[\sqrt{2},\sqrt{3}] - \mathbb{Q}[\sqrt{2}] \\ 1/5\ , & x \in \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}] - \mathbb{Q}[\sqrt{2},\sqrt{3}] \\ 1/7\ , & x \in \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7}] - \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5}] \\ 0\,, & x \in \mathbb{Q}[\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7},\dots] = \mathbb{R} \end{cases}$$

Since for each $t \in [0,1]$, A_t is a *G-domain* as the follows:

$A_0 = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \dots] = \mathbb{R}$, $A_{1/7} = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}]$, $A_{1/5} = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$, $A_{1/3} = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$, $A_{1/2} = \mathbb{Q}[\sqrt{2}]$ *and* $A_1 = \mathbb{Q}$.Hence *A* is a *fuzzyG-subdomain* of $\mathbb{R}$.

Lemma 2.3. If *A* be a *fuzzy G-subdomain* of *D* with the quotient fuzzy subfield *K* for D_A, then For $0 \neq u \in D$ we have:
$K = D_A[u^{-1}]$

Lemma 2.4. *Let A be a fuzzy subdomain of D and u be an element located on a domain containing*D_A *, if* $D_A[u]$ *is a G-subdomain, then: A is a fuzzy G-subdomain of D and u is algebraic on* D_A *.*

Theorem 2.5. *A is a fuzzy G-subdomain of D if and only if for each* $t \in [0,1]$, A_t *is a G-subdomain of* D_A.

Proof. Let *A* be a fuzzy *G-subdomain* of *D* and $t \in [0,1]$, since $A(0) \geq A(x),\ \forall x \in D_A$, then: $0 \in A_t, therefore\ A_t \neq \emptyset$ .Now let $x, y \in A_t, since\ A(x) \geq t,\ A(y) \geq t\ and\ A$ is a fuzzy subdomain of *D*, then:

$$A(xy) \geq inf\{A(x), A(y)\} = A(x) \wedge A(y) \geq t \wedge t = t$$
$$\Rightarrow A(xy) \geq t \Rightarrow xy \in A_t$$

Furthermore:

$$A(x - y) \geq \inf\{A(x), A(y)\} = A(x) \wedge A(y) \geq t \wedge t = t$$
$$\Rightarrow A(x - y) \geq t \Rightarrow x - y \in A_t$$

Therefore A_t is a fuzzy subdomain of D_A and so it's a fuzzy subdomain of *D*.
In addition, for the quotient field of *K* related to D_A, we have $K = D_A[u^{-1}]$, $\exists\ 0 \neq u \in K$ and since A_t is a subdomain of D_A, so

$$A_t \leq D_A \Rightarrow A_t[u^{-1}] \leq D_A[u^{-1}] = K$$

Since by Lemma "2.3" *u* is algebraic on A_t, then A_t is a G-subdomain of D_A.
Conversely, let A_t be a G - subdomain of D_A, $\forall t \in A(D_A)$.
Since $0 \in A_t$, then $A(0) \geq t$.
Let $x, y \in D_A$ and let $A(x) = t_1,\ A(y) = t_2\ and\ t_3 = t_1 \wedge t_2\ so\ x, y \in D_{A_{t_3}}$, therefore $t_3 \leq A(0)$, therefore $D_{A_{t_3}}$ is a subdomain of *D* and hence $xy \in D_{A_{t_3}}$.
Now:

$$A(xy) \geq t_3 = t_1 \wedge t_2 = A(x) \wedge A(y)$$

Since *A* has the G- structure property, therefore *A* is a fuzzy *G*-subdomain of D_A and hence *A* is a fuzzy *G*-subdomain of *D*. □

Theorem 2.6. *Let A is a fuzzy G-subdomain of D with the quotient field K of* D_A *and let B be a fuzzy subring, such that:*

$$D_A \subseteq D_B \subseteq K$$

Then B is a fuzzy G-subdomain of D.

Proof. Since $K = D_A[u^{-1}]$ for some $u \in K$, then $K = D_A[u^{-1}] = D_B[u^{-1}]$, therefore *B* is a fuzzy *G*-subdomain of *D*. □

Theorem 2.7. *Let* $A, B \in F(D)$ *be arbitrary members, if* $A \subset B\ and\ D_B$ *is algebraic on*D_A*and*D_B *as a subring above* D_A *is finitely generated, then:*
A is a fuzzy G-subdomain ofD if and only if B is a fuzzy G-subdomain of D.

Proof. Let K , L be the quotient fuzzy subfield of D_A , D_B. Suppose first that A is a fuzzy G-subdomain, say $K = D_A[u^{-1}]$. Then $D_B[u^{-1}]$ is a fuzzy subdomain algebraic over the quotient field of K, hence itself is a field, necessarily equal to L.Thus B is an fuzzyG-subdomain of D.

*Conversely,*assume that B is a fuzzy G-subdomain of D, $L = D_B[v^{-1}]$ and $D_B = D_A[w_1, w_2, \dots, w_k]$. The elements $v^{-1}, w_1, w_2, \dots, w_k$ are algebraic over D_A and consequently satisfy equations with coefficients in D_A which lead off, say:

$$av^{-m} + \dots = 0$$

$$b_i w_i^{n_i} + \dots = 0 \ (i = 1, \dots, k)$$

Adjoin $a^{-1}, b_1^{-1}, \dots, b_k^{-1}$ to D_A, obtaining a subring D_1 between D_A and K. The field L is generated over D_A by $w_1, \dots, w_k, v^{-1}$. Of course these elements generate L over D_1. Now over D_1 we have arranged that $w_1, \dots, w_k, v^{-1}$ are integral. Hence L is integral over D_1. Therefore, D_1 is a field,necessarily K. So K is a finitely generated ring over $D_A$and therefore A is anfuzzy G-subdomain of D, as required. □

Theorem 2.8. *Let A be a fuzzy subdomain of D and u is an element of a larger of D_A , if $D_A[u]$ is a G-domain, then \$u\$ is algebraic over D_A and A is a fuzzy G-subdomain of D.*

Proof. Let $T_A = D_A[u]$ is a G-domain if we define $L_A = T_A[v^{-1}]$, then u, v^{-1} are algebraic on D_A, because for each $\alpha \in T_A$, $t \in D_A[u]$, in especial case we have $0 \in T_A$, then:

$$0 = a_0 + a_1 u + \dots + a_n u^n$$

So u is algebraic on D_A .

On the other hand T_A as a ring on D_A is finitely generated, therefore by Theorem "2.3" D_A is G-domain and hence A is fuzzy G-subdomain of D. □

Theorem 2.9. *Let A and B are two fuzzy subdomains of D such that $D_A \subseteq D_B$ and D_B finitely generated as a ring over D_A,and D_B is not algebraic over D_A, then B is not a fuzzy G-subdomain of D.*

Proof. Let $D_B = D_A[\alpha_1, \alpha_2, \dots, \alpha_n]$ and let $\alpha_{m+1}, \dots, \alpha_n$ are not algebraic over D_A. By Theorem"2.4" $D_A[\alpha_{m+1}]$ is not a G-domain.

Put $D_{A_0} = D_A[\alpha_1, \alpha_2, \dots, \alpha_m][\alpha_{m+1}], then D_{A_0}$ is not a G-domain. Again, consider α_{m+2}, if α_{m+2} is not algebraic over D_{A_0} then $D_{A_1} = D_{A_0}[\alpha_{m+2}]$ is never an G-domain and if α_{m+2} is not algebraic over D_{A_0},then D_{A_1} is not a G-domain. However D_{A_1} is not a G-domain. By repetition this process we obtain that D_B is not a \$G-domain\$ and therefore B is not a fuzzy G-subdomain of D. □

Lemma 2.10.*Let R be a ring and A is a fuzzy subring of R,*

1) If M is an ideal of $R_{A[x]}$ and satisfying in $M \cap R_A = 0$. Let ube the image of x under the canonical homomorphism $R_A[x] \longrightarrow \frac{R_A[x]}{M}$,then:

$$\frac{R_A[x]}{M} \simeq R_A[u]$$

In addition, if M is maximal in $R_A[x]$, then the $R_A[u]$ is a field.

2) In general form, if M is a maximal ideal of $\mathcal{R}_A = R_A[x_1, x_2, \dots, x_n, \dots]$ satisfy $M \cap R_A = 0$, and if u_i is the image of $x_i + M$ for each $i \in \aleph_0$ under the canonical homomorphism

$$\frac{\mathcal{R}_A}{M} \longrightarrow \mathcal{R}_A \quad , \quad \forall\, i \in \aleph_0$$

Therefore

$$\frac{\mathcal{R}_A}{M} \simeq R_A[u_1, u_2, \ldots, u_n, \ldots]$$

Proof.1) At first, the note $R_A \cap M = 0$ requires that $R_A \simeq \{a + M: a \in R_A\}$,which is a subring of $R[x]/M$ (say: S_A). Now,

$$R_A[u] \simeq S_A[x + M] = \{\sum_{i=0}^{n} s_i(x + M)^i : s_i \in S_A, n \in \aleph_0\}$$

$$= \{\sum_{i=0}^{n} (a_i + M)(x + M)^i : a_i \in R_A, n \in \aleph_0\}$$

$$= \{\sum_{i=0}^{n} (a_i + M)(x^i + M) : a_i \in R_A, n \in \aleph_0\}$$

$$= \{\sum_{i=0}^{n} (a_i x^i + M) : a_i \in R_A, n \in \aleph_0\}$$

$$= \left\{\sum_{i=0}^{n} (f(x) + M) : f(x) \in R_A[x]\right\} = \frac{R_A[x]}{M}$$

2) By similar argument that is expressed in part "1" and this fact that any element of $\mathcal{R}_A$ is of the form:

$$F(\bar{X}) = \{\sum_{(i_1,i_2,\ldots,i_n)\in\Lambda} a_{i_1,i_2,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n} \;, for\ some\ finite\ subset\ \Lambda \subseteq \aleph_0^n\}$$

We have:

$$R_A[u_1, u_2, \ldots, u_n, \ldots] \simeq S_A[x_1 + M, x_2 + M, \ldots, x_n + M, \ldots]$$

$$= \{\textstyle\sum_{(i_1,i_2,\ldots,i_n)\in\Lambda} s_{i_1,i_2,\ldots,i_n} (x_1 + M)^{i_1} \ldots (x_n + M)^{i_n} \;, s_{i_1,i_2,\ldots,i_n} \in S_A\}$$

$$= \{\textstyle\sum_{(i_1,i_2,\ldots,i_n)\in\Lambda} (a_{i_1,i_2,\ldots,i_n} + M)(x_1 + M)^{i_1} \ldots (x_n + M)^{i_n} \;, a_{i_1,i_2,\ldots,i_n} \in R_A\}$$

$$= \{\textstyle\sum_{(i_1,i_2,\ldots,i_n)\in\Lambda} (a_{i_1,i_2,\ldots,i_n} + M)\left(x_1^{i_1} + M\right) \ldots \left(x_n^{i_n} + M\right) \;, a_{i_1,i_2,\ldots,i_n} \in R_A\}$$

$$= \left\{\sum_{(i_1,i_2,\ldots,i_n)\in\Lambda} a_{i_1,i_2,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n} \;, for\ some\ finite\ subset\ \Lambda \subseteq \aleph_0^n \;, a_{i_1,i_2,\ldots,i_n} \in R_A\right\}$$

$$= \{F(\bar{x}) + M, where\ F(\bar{x}) is\ a\ typical\ element\ of \mathcal{R}_A\} = \frac{\mathcal{R}_A}{M}$$

Which is proved the second part. □

Theorem 2.11. *The fuzzy subset A of an integral domain D is a fuzzy G-subdomain of D if and only if there exists in the polynomial ring $D_A[x]$ an ideal M which is maximal and satisfies M \cap $D_A = 0$.*

Proof. Let A be a fuzzy G-subdomain of D, we have: $K = D_A(u^{-1})$, where K is a quotient field of D_A. Now we define the ring homomorphism as the following:

$$\phi: D_A[x] \longrightarrow K = D_A[u^{-1}]$$
$$x \mapsto u^{-1}$$

it is obvious that the image is all of K, so the kernel M is maximal.
Since the homomorphism is one to one on R, therefore we have $M \cap D_A = 0$.
Conversely, let M be maximal ideal in $D_A[x]$ and satisfying in $M \cap D_A = 0$. Denote the image of x in the natural homomorphism $D_A[x] \longrightarrow D_A[x]/M$ by ν . Then by lemma"2.3" $D_A[\nu]$ is a field and therefore by Theorem "2.4" D_A is a G-domain and hence A is a fuzzy G-subdomain of D. □

3. Properties of Fuzzy G-Ideals

Theorem 3.1. *Let* $\{A_t | t \in [0,1]\}$ *is a collection of ideals in R such that:*
i) $R = \bigcup_{t\in[0,1]} A_t$
ii) $t_1 \geq t_2 \Leftrightarrow A_{t_1} \subseteq A_{t_2}, \quad \forall\, t_1, t_2 \in [0,1]$.
Then for each subset S of R in which, $S(x) = sup\{t_i \in [0,1]: x \in A_{t_i}\}$, *for all* $x \in R$,*S is a fuzzy ideal of R*

Proof. It is sufficient that we prove S_t is an ideal of R, for each $t \in [0,1]$ with $S_t \neq \emptyset$.
Let $t \in [0,1]$. We have two cases:
1)$t = sup\{\, t \in [0,1] | r < t\}$,
2)$t \neq sup\, \{t \in [0,1] | r < t\}$.
Case (1) implies that for each $r \in [0,1]$ and $r < t, x \in S_t$ this equivalent with

$$x \in A_r\,, \forall\, r < t\,, r \in [0,1]\,. i.e.,$$

$$x \in \bigcap_{r<t_{r\in[0,1]}} A_r$$

Hence

$$S_t = \bigcap_{r<t_{r\in[0,1]}} A_r$$

Which is an ideal of R.
Case (2) implies that, there exists $\epsilon > 0$ such that $[t-\epsilon, t] \cap [0,1] = \emptyset$.
If $x \in \bigcup_{r<t_{r\in[0,1]}} A_r$, $then,\$\, x \in A_r\ for\ some\ r \geq t$. It follows that $S(x) \geq r \geq t$,so that $x \in S_t$. That is $x \in \bigcup_{r<t_{r\in[0,1]}} A_r < S_t$.
Conversely, if $x \notin x \in \bigcup_{r<t_{r\in[0,1]}} A_r$, then $x \notin A_r$ for all $r \geq t$.
This implies that $x \notin A_r$,$\forall r > t - \epsilon$,in other word, if $x \in A_r$ then $r \leq t - \epsilon$ .Thus $S(x) \leq t - \epsilon$ and so $x \notin S_t$. Therefore:

$$S_t = \bigcup_{r<t_{r\in[0,1]}} A_r$$

Hence S_t is an ideal of R. □

Definition 3.2. The fuzzy prime ideal P in commutative domain D is called a *fuzzy G- ideal* if *R/P* is a fuzzy *G*-subdomain.

Theorem 3.3. *P is a fuzzy G-ideal if and only if for each* $t \in [0,1], P_t$ *is an G-ideal of D.*

Proof. Let P is a fuzzy G-ideal of D and $t \in [0,1]$ be arbitrary. Since $P(0) \geq P(x)$,for each $x \in D$, then $0 \in P_t$ and so $P_t \neq \emptyset$. Now let $x, y \in P_t$,since $P(x) \geq t, P(y) \geq t$ and P is also a fuzzy G-ideal of D then:

$$P(xy) \geq inf\{P(x), P(y)\} = P(x) \wedge P(y) \geq t \wedge t = t$$

$$\Rightarrow \mathrm{P(xy)} \geq \mathrm{t} \Rightarrow \mathrm{xy} \in \mathrm{P_t}$$

In addition:

$$P(x-y) \geq inf\{P(x), P(y)\} = P(x) \wedge P(y) \geq t \wedge t = t$$

$$\Rightarrow \mathrm{P(x-y)} \geq \mathrm{t} \Rightarrow \mathrm{x-y} \in \mathrm{P_t}$$

Therefore for each $t \in [0,1]$, P_t is an ideal of D.
Now since P_t is an ideal of D and P_t is also a subdomain of D and for some $0 \neq u \in K$, $K = D[u^{-1}]$So:

$$P_t \leq D \Longrightarrow P_t[u^{-1}] \leq D[u^{-1}] = K$$

Therefore by Theorem "3.1' P_t is a G-ideal of D.
Conversely, Let P_t is an G-ideal of D , for each $t \in P(D) \cup \{b \in [0,1]: b \leq P(0)\}$,since $0 \in P_t$, $\forall\, t \in P(D)$, therefore:

$$P(0) \geq t \;, \forall\, t \in P(D)$$

Let $x, y \in D$ and suppose that $P(x) = t_1$, $P(y) = t_2$ *and* $t_3 = t_1 \wedge t_2$, then $\mathrm{x, y} \in \mathrm{P_{t_3}}$, therefore$\mathrm{P_{t_3}}$is an ideal of D and so $xy \in P_{t_3}$.
Now we have:

$$\mathrm{P(xy)} \geq \mathrm{t_3} = \mathrm{t_1} \wedge \mathrm{t_2} = \mathrm{P(x)} \wedge \mathrm{P(y)}$$

In addition Since P has the G-structure property, therefore P is an *G-fuzzy ideal* of D. □

Definition 3.4. The fuzzy ideal P of the ring R is called a fuzzy primary ideal if for each $a, b \in R$, $P(ab) \geq P(a) then$: $P(b^n) \geq P(ab)$, $\exists n \in Z^+$.

Theorem 3.5. *Let P is an ideal of the ring of R, the Characteristic function* λ_P *is a fuzzy primary ideal of R if and only if P is a primary ideal of R.*

Proof. Let P be a primary ideal of R , suppose that for each $a, b \in R$ we have:
$\lambda_P(ab) > \lambda_P(a), since \lambda_P(x)$ either zero or 1, therefore by the condition of
$\lambda_P(ab) > \lambda_P(a)$ we have:
$\lambda_P(a) = 0$ and$\lambda_P(ab) = 1$

Therefore: ab ∈ P and a ∉ P . Since P is primary, so $b^n \in P, \exists n \in Z^+$.Hence:
$\lambda_P(b^n) = 1 = \lambda_P(ab)$ and then λ_P is a fuzzy primary ideal of R.
Conversely, Let λ_P is a fuzzy primary ideal and let $ab \in P$, $a \notin P$ *then* $\lambda_P(ab) = 1$and $\lambda_P(a) = 0$, therefore:$\lambda_P(ab) > \lambda_P(a)$.
Since λ_P is a fuzzy primary, then there exists some positive integer number n such that:
$\lambda_P(b^n) > \lambda_P(ab)$, $so\ \lambda_P(b^n) = 1$.This show that $b^n \in P$ and hence P is a primary ideal of R. □

Theorem 3.6. *Let P is a fuzzy ideal of the ring of R, then P is a fuzzy primary ideal if and only if* $\forall t \in [0, P(0)]$, $P_t = \{x \in R: P(x) \geq t\}$*is a primary ideal of R.*

Proof. It is obvious that for every fuzzy ideal P, the P_t , $\forall t \in [0, P(0)]$ is an ideal of R. Now let P is a fuzzy primary ideal of R and suppose that $ab \in P_t$, $a \notin P_t$, then: $P(ab) \geq t$, $P(a) < t$ and so:

$$P(ab) > P(a)$$

Since P is a fuzzy primary ideal, so there exists some positive integer number n such that $P(b^n) \geq P(ab) \geq t$.Therefore: $b^n \in P_t$ and so P_t is a primary ideal.
Conversely,Let P_t , $\forall t \in [0, P(0)]$ is a primary ideal of R, suppose that:

$$P(ab) > P(a) \quad , \quad \forall a, b \in R$$

If we define t = P(ab) , therefore:

$$t \in [0, P(0)] , ab \in P_t \; and t > P(a).$$

So we have: $a \notin P_t$
Therefore $\exists n \in Z^+ \; such \; that$: $b^n \in P_t$. Hence:

$$P(b^n) \geq t = P(ab)$$

This implies that P is a fuzzy primary ideal. □

4. On Fuzzy G-Primary Ideals

Remark 4.1. [1] *1) Let D is a domain , The ideal I of D is called a* $\sqrt{G} - ideal$ *, if* $D/\sqrt{I}$ *is a G-domain (i.e.,* $\sqrt{I}$ *is a G-ideal).*
2) Let D be a domain, The primary ideal Q of D is called a G-primary ideal if Q is a $\sqrt{G} - ideal.$

Definition 4.2. Let D is a domain, the fuzzy ideal I of D is called a fuzzy $\sqrt{G} - ideal$, if $D/\sqrt{I}$ is an fuzzy G-domain (i.e., $\sqrt{I}$ is a fuzzy G-ideal).

Definition 4.3. Let D is a domain, the primary ideal Q of D is called an fuzzy G-primary ideal, if Q is a fuzzy $\sqrt{G} - ideal$.

Lemma 4.4. *Every fuzzy G-ideal of R is also an fuzzy G-primary ideal of R.*

Proof. If P is an fuzzy G-ideal of R then:

$$\sqrt{\mathrm{P}} = \mathrm{P} \Longrightarrow \mathrm{R}/\sqrt{\mathrm{P}} = \mathrm{R}/\mathrm{P}$$

So, $R/\sqrt{P}$ is also an fuzzy G-domain, therefore $\sqrt{P}$ is an fuzzy G-ideal and hence P is a fuzzy G-primary ideal. □

Remark 4.5. [1] Let R be an integral domain and I , J are two ideals of R, if $I \subseteq J$ then:

$$\sqrt{\left(\frac{J}{I}\right)} = \sqrt{(J)}/I.$$

Theorem 4.6.*Let* $\Phi: R \longrightarrow T$ *is a fuzzy ring epimorphism and R , T are two fuzzy integral domains. If Q is a fuzzy primary G-ideal of T , then* Q^c *is a fuzzy G-primary ideal of R.*

Proof. Let Q be a fuzzy G-primary ideal of T and $\sqrt{Q} = P$, then:

$T/\sqrt{Q} = T/P$ is a fuzzy G-domain , since P is a fuzzy prime ideal of T so P^c is also a fuzzy prime ideal of R and since $P^c = \left(\sqrt{Q}\right)^c = \sqrt{Q^c}$, then $\sqrt{Q^c}$ is a fuzzy prime ideal of R , $R/\sqrt{Q^c}$ is a fuzzy domain, therefore $\sqrt{Q^c} = P^c$ is a fuzzy prime ideal of R.
Now since P is a fuzzy G-ideal of T and R/P^c is isomorphic by a fuzzy subdomain of *T/P* ,then:

$$\frac{R}{P^c} \simeq T^*, \quad \exists T^* \leq_{subfield} T/P$$

Now since T* is a fuzzy subdomain , then R/P^c is a fuzzy G-subdomain , therefore $\left(\sqrt{Q}\right)^c$ is a fuzzy G-ideal of R and hence Q^c is a fuzzy G-primary ideal of R. □

Corollary 4.7. *Let $\Phi: R \longrightarrow S^{-1}R$ is the canonical homomorphism defined by $\Phi(r) = r/I$. The set of all fuzzy G-primary ideals of R which in satisfying the condition: $P \cap S = \emptyset$ is equals to the set of all fuzzy G-primary ideals on the form of $PS^{-1}R$ of $S^{-1}R$.*

Proof.(⇒) Let Q is any fuzzy G-primary ideal of R such that it is satisfying on the condition of $Q \cap S = \emptyset$
Firstly: by Theorem "37.5" of [14], since Q is a primary ideal of R so Q^c is also a primary ideal of $S^{-1}R$.
Secondly: since Q is a fuzzy ideal of R so Q^c is also a fuzzy ideal of $S^{-1}R$, therefore Q^c is a fuzzy primary ideal of $S^{-1}R$, since Q on R is a fuzzy primary ideal, then Q^e is also in $S^{-1}R$ at first is a primary ideal and secondly is a fuzzy ideal of $S^{-1}R$. Now we have:

$$\left(\sqrt{Q}\right)^e = \sqrt{Q^e}$$

Now since $R/P = R/\sqrt{Q}$ is a fuzzy G-subdomain and we have:

$$\frac{S^{-1}R}{PS^{-1}R} \simeq \tilde{S}^{-1}(R/P)$$

Where $\tilde{S} = \{\tilde{s} \in R/P: s \in S\}$ and $\tilde{S}^{-1}(R/P)$ under ring isomorphism is a fuzzy overring of R/P located between R/P and its quotient field , therefore by Theorem 20 of [8] $\bar{S}^{-1}(R/P)$ is a G-domain and so totally, it is a fuzzy G-subdomain.
Hence $\sqrt{Q^e} = \left(\sqrt{Q}\right)^e = P^e$ is a fuzzy G-ideal of $S^{-1}R$ (i.e., Q^e is a fuzzy G-primary ideal of $S^{-1}R$).
(⇐) By Remark "1.2" it is obvious. □

Corollary 4.8.*If $Q_1, Q_2, \ldots, Q_n$ are fuzzy G-primary ideals of R, then $\bigcap_{i=1}^{n} Q_i$ is a fuzzy G-primary ideal of R.*

Proof. Firstly, it is obvious that we have:

$$\sqrt{\bigcap_{i=1}^{n} Q_i} = \bigcap_{i=1}^{n} \sqrt{Q_i}$$

Where is satisfying on fuzzy G-primary ideals.
Now let:

$$\sqrt{\mathrm{Q_i}} = \mathrm{P_i}, \forall \mathrm{i} \in \{1,2,\ldots,\mathrm{n}\}$$

Since:

$$\bigcap_{i=1}^{n} P_i \subseteq \mathrm{P_j}, \qquad \forall\ \mathrm{j} \in \{1,2,\ldots,\mathrm{n}\}$$

So we have the following isomorphism:

$$\frac{(\mathrm{R}/\bigcap_{i=1}^{n} P_i)}{(\mathrm{P_j}/\bigcap_{i=1}^{n} P_i)} \simeq \frac{\mathrm{R}}{\mathrm{P_j}}, \forall \mathrm{j} \in \{1,2,\ldots,\mathrm{n}\ \}$$

Since on $(\mathrm{R}/\bigcap_{i=1}^{n} P_i)$ the intersection of nonzero fuzzy prime ideals are nonzero, then $(\mathrm{R}/\bigcap_{i=1}^{n} P_i)$ is a fuzzy G-subdomain and therefore $\bigcap_{i=1}^{n} P_i$ is a fuzzy G-ideal and hence $\bigcap_{i=1}^{n} Q_i$ is a fuzzy G-primary ideal of R. □

It is suggested that further research in this direction is likely going to reveal additional properties of fuzzy G-type ideals associated to fuzzy subdomains and thus contribute to our understanding of how such structures defines on the underlying fuzzy G-type subdomains.

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