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GENERALIZED DOUBLE SEQUENCE SPACE DERIVED BY USING SUMMABILITY MATRIX AND DOUBLE SEQUENCE OF MODULUS FUNCTIONS OVER n-NORMED SPACES

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ABSTRACT

In this paper, we introduce some double sequence spaces derived by using four dimensional summability matrix and double sequence of modulus functions $\mathcal{M} = (M_{kl})$ *over n-normed spaces, We study some topological properties and a few inclusion relations between these spaces.*

KEYWORDS: Double sequence space, Orlicz function, P-convergence, summability matrix

Mathematical Subject Classification: 40A05 40C05 46A45

INTRODUCTION

The concept of 2- normed space was initially developed by Gähler [1] in the mid of 1960's, while that of n- normed spaces one can see in Misiak [2]. Since then, many others have studied this concept and obtained various results, see Gunawan ([3], [4]) and Gunawan and Mashadi [5] and various references therein.

Let $n \in \mathbb{N}$ and X be a linear spaces over the field K, where K is the field of real or complex numbers of dimension $d \, where \, d \geq n \geq 2$. A real valued function $\|., \dots, .\|$ on X^n satisfying the following four conditions:

1 $\|x_1, x_2, \dots, x_n\| = 0, if \ and \ only \ if x_1, x_2, \dots, x_n$ are linear dependent in X.

2 $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation

3 $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|, for \ any \ \alpha \in K$

4 $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

Is called a n- norm on X, and the pair $(X, \|., \dots,. \|)$ is called a n-normed space over the field K. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n- norm $\|x_1, x_2, \dots, x_n\|_E$= the volume of the n-dimensional Parallelopiped spanned by the vectors $x_1, x_2, \dots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{i\,j})|$$

Where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$.

Let $(X, \|x_1, x_2, \dots, x_n\|)$ be a n- normed space of dimension $d \geq n \geq 2 \ and \ \{a_1, a_2, \dots, a_n\}$ be linearly independent set in X. Then the following function $\|., \dots, .\|_\infty \ on \ X^{n-1}$ defined by

$\|x_1, x_2, \dots, x_n\|_\infty = max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1,2, \dots, n\}$ defines on (n-1)- norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|., \dots, .\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0, for \ every \ z_1, \dots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|., \dots, .\|)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0, for \ every \ z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n- norm. Any complete n-normed space is said to be n- Banach space.

The initial works on double sequence is found in Bromwich [6]. Later on, it was studied by Hardy [7], Moricz [8], Moricz and Rhoades[9], Tripathy ([10] [11]), Başarir and Sonalcan [12] and many others. Hardy [13] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser[14] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [16] have recently studied the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly cesăro summable double sequences, Mursaleen [9] and Mursaleen and Edely [15] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduce M-Core for double sequences and determined those four dimensional matrices transforming every bounded sequence $x = (x_{mn})$ into one whose core is a subset of the M-Core of x. More recently, Altay and Başar [17] have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r \ and \ BV$ of double sequence consisting of all double series whose sequence of partial sums are in the space $\mathcal{M}_u, \mathcal{M}_u(t), C_p, C_{bp}, C_r \ and \ \mathcal{L}_u$ respectively and also examined some properties of those sequence spaces as well as the α −duals of these spaces $BS, BV, CS_{bp} \ and \ the \ \beta(v) - duals$ of the spaces $CS_{bp} \ and \ CS_r$ of double series. Now, recently Başar and Sever [18] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known ℓ_q of single sequences and determined some properties of the $\mathcal{L}_q$. The class of sequences which are strongly cesaro summble with respect to a modulus function was introduced by Maddox [19] as an extension of the definition of strongly cesăro summable sequences. Cannor [20] further extended this notion to strongly A-summability with respect to a modulus where $A = (a_{nk})$ is a non negative regular matrix, using the definition Cannor established connection between strong A-summability, strong A-summability with respect to a modulus and A-Statistical convergence. In 1900, Pringsheim [21] presented a definition for convergence of double sequences. Following Pringsheim work, Hamilton and Robinson [5] and [22], respectively presented a series of necessary and sufficient conditions on the entries of $A = (a_{m,n,k,l})$ that ensure the presentation of Pringsheim type convergence on the following transformation of double sequences.

$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{kl}$.

Throughout this paper the four dimensional matrices and double sequences are of real valued entries unless otherwise specified. Let s'' denotes the set of all double sequences of complex numbers. By the convergence of a double sequence we mean the convergence in Pringsheim sense i.e a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by P-limit x= L) provided that given $\epsilon > 0$ *there exists* $n \in \mathbb{N}$, *such that* $|x_{kl} - L| < \epsilon$, *whenever* $k, l > n$ (see[21]). We shall write more briefly as P-convergent. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k, l.

The notion of difference sequence spaces was introduced by Kizmaz [23], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. This notion was further generalized by Et and çolak [24] defined the sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let m, n a non negative integers we have the following spaces:

$$Z(\Delta_t^v) = \{x = (x_k) \in \omega : (\Delta_t^v x_k) \in Z\}$$

For $Z = c, c_0, and\ l_\infty$ where

$$\Delta_t^v x = (\Delta_t^v x_k) = (\Delta_t^{v-1} x_k - \Delta_t^{v-1} x_{k+1}), \text{ and } \Delta_t^0 x_k = x_k$$

For all $k \in N$, which is equivalent to binomial representation

$$\Delta_t^v x_k = \sum_{i=0}^{v} (-1)^i \binom{v}{i} x_{+ti}$$

It was proved that the generalized sequence space $Z(\Delta_t^v)$, where $Z = \ell_\infty, c$ or c_0, is a Banach space with norm defined by

$$\|x\|_{\Delta_t^v} = \sum_{i=1}^{v} |x_i| + sup|\Delta_t^v x_k|.$$

Taking $t = 1$, we get the spaces which were introduced and studied by Et and çolak [24].

Taking $t = v = 1$, we get the spaces which were introduced and studied by Kizmaz [23].

Let X be a real or complex linear space p be a function from X to the set $\mathbb{R}$ of real numbers. Then the pair (X, p) called a paranormed space and p is a paranorm for X if the following axioms are satisfied:

1 $p(\theta) = 0$

2 $p(-x) = p(x)$

3 $p(x + y) \leq p(x) + p(y)$ and

4 Scalar multiplication is continuous, that is $|\alpha_n - \alpha| \to 0 and\ p(x_n - x) \to 0, imply\ p(\alpha_n x_n - \alpha x \to 0,$ for all $\alpha\ in \mathbb{R}$ and x in X. where θ is the zero vector in the space X.

A paranorm p for which $p(x) = 0, implies\ x = \theta$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [25])

A modulus function is a function $\mathcal{M}: [0, \infty) \to [0, \infty)$ such that

1 $\mathcal{M}(x) = 0\ if\ and\ only\ if\ x = 0.$

2 $\mathcal{M}(x + y) \leq \mathscr{M}(x) + \mathscr{M}(y), \forall\ x \geq 0, y \geq 0.$

3 $\mathcal{M}$ is increasing

4 $\mathcal{M}$ is continuous from right at 0.

It follows that $\mathcal{M}$ must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $\mathcal{M}(x) = \frac{x}{x+1}$, then $\mathcal{M}(x)$ is bounded. If $\mathcal{M}(x) = x^p, 0 < p < 1$, then the modulus $\mathcal{M}(x)$ is unbounded. Modulus function has been discussed in ([27], [28], [29]) and reference therein.

Let $A = (a_{m,n,k,l})$ denotes a four dimensional summability matrix that maps the complex double sequence x into the double sequence Ax where mn^{th} term of Ax is as follows:

$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{kl}$.

Let $\mathcal{M} = (M_{kl})$ be a sequence of modulus functions and $A = \left(a_{m,n,k,l}\right)$ be a non negative four dimensional matrix of real entries with

$$\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$$

Let $p = (p_{k,l})$ be bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers

Sharma and Esi [26] have defined and studied the double sequence defined by a sequence of modulus functions over n- normed spaces and discussed the topological properties and inclusion relations between these spaces.

In this paper we define the following sequence spaces: Let $s \geq 0$

(i). $\omega_0^{''}(\Delta_t^v, A, \mathscr{M}, u, p, s, \|., \dots, .\|)$

$$= \left\{ x \in s^{''}: \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\| \frac{\Delta_t^v x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_{kl}} \right] = 0, \rho > 0 \right\}.$$

(ii). $\omega^{''}(\Delta_t^v, A, \mathscr{M}, u, p, s, \|., \dots, .\|)$

$$= \left\{ x \in s^{''} : \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\| \frac{\Delta_t^v x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_{kl}} \right] = 0, for\ some\ L, \rho > 0 \right\}$$

(ii). $\omega_\infty^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$

$$= \left\{ x \in s^{''} : \sup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\| \frac{\Delta_t^v x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_{kl}} \right] < \infty, \rho > 0 \right\}$$

The following inequality will be used throughout the paper.

Let $p = (p_{kl})$ be a sequence of positive real numbers with $0 \leq p_{kl} \leq \sup_{kl} p_{kl} = H\ and\ k = \max(1, 2^{H-1})$ Then

$$|a_{kl} + b_{kl}|^{p_{kl}} \leq K\{|a_{kl}|^{p_{kl}} + |b_{kl}|^{p_{kl}}\} \qquad (1.1)$$

For all $k, l\ and\ a_{kl} and\ b_{kl} \in \mathbb{C}$. Also $|a|^{p_{kl}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$

2. MAIN RESULTS

The following theorems will be proved in this section.

Theorem 2.1 Let $\mathcal{M} = (M_{kl})$ be a sequence of modulus functions, $A = \left(a_{m,n,k,l}\right)$ be a non-negative matrix such that $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$, $p = (p_{k,l})$ be bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers, $s \geq 0$,the spaces $\omega_0^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$, $\omega^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ and $\omega_\infty^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ are linear spaces over the field of complex numbers $\mathbb{C}$.

Proof. Let $x = (x_{kl}), y = (y_{kl}) \in \omega_0^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] = 0, \rho_1 > 0 \quad (2.1)$$

$$P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] = 0, \rho_2 > 0 \quad (2.2)$$

Define $\rho_3 = max\left(2|\alpha|\rho_{1,}, 2|\beta|\rho_2\right)$. Since $\mathcal{M} = (M_{kl})$ is increasing, continuous and so by using inequality (1.1), we have

$$P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_{t(\alpha}^v x_{k,l} + \beta y_{kl})}{\rho_3}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$\leq P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\alpha \Delta_t^v x_{k,l} + \beta \Delta_t^v y_{kl}}{\rho_3}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$\leq KP - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$+KP - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$\to 0.$

Thus $\alpha x + \beta y \in \omega_0^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$. This proves that the space is a linear space. Similarly, we can prove that $\omega^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ and $\omega_\infty^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ are linear spaces.

Theorem 2.2 Let $\mathcal{M} = (M_{kl})$ be a sequence of modulus functions, $A = \left(a_{m,n,k,l}\right)$ be a non-negative matrix such that $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$, $p = (p_{k,l})$ be bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers, $s \geq 0$,the spaces $\omega_0^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$, $\omega^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ and $\omega_\infty^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ are complete topological linear spaces.

Proof. Let $(x_{k,l}^r)$ be a Cauchy sequence in $\omega_0^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$. Then, we write $g(x^r - x^h) \to 0,\ as\ r, h \to \infty, \forall m, n$ we have

$$P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{kl}^r - \Delta_t^v x_{kl}^h}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] = 0 \qquad (2.3)$$

Thus for each fixed $k\ and\ l\ as\ r, h \to \infty, since\ A = (a_{m,n,k,l})$ is non-negative we are granted that

$$(kl)^{-s} u_{k,l} \left[M_{k,l} \left(\left\|\frac{\Delta_t^v x_{kl}^r - \Delta_t^v x_{kl}^h}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] \to 0$$

And by continuity of $\mathcal{M} = (M_{kl})$ and $(x_{k,l}^r)$ is a Cauchy sequence in $\mathbb{C}$ for each fixed k and l. Since $\mathbb{C}$ is complete as $h \to \infty$, we have $x_{kl}^r \to x_{kl}\ for\ each\ (k, l)$. Now from equation (2.3), we have for $\epsilon > 0$, there exists a natural number N such that

$$\sum_{k,l=1,1,r,h>N}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{kl}^r - \Delta_t^v x_{kl}^h}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] < \epsilon \qquad (2.4)$$

For all m,n, since for any fixed natural number M we have from equation (2.4)

$$\sum_{k,l \leq M, r,h>N}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{kl}^r + \Delta_t^v x_{kl}^h}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] < \epsilon$$

$For\ all\ m, n\ by\ letting\ h \to \infty$ in the above expression we obtain

$$\sum_{k,l \leq M, r>N}^{\infty,\infty} (kl)^{-s} u_{k,l} \left[a_{m,n,k,l} M_{k,l} \left(\left\|\frac{\Delta_t^v x_{kl}^r + \Delta_t^v x_{kl}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right] < \epsilon$$

Since M is arbitrary, by letting $M \to \infty$, we obtain

$$\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{kl}^r+\Delta_t^v x_{kl}^h}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]<\epsilon$$

For all $m,n. Thus\ g(x^r-x)\to 0 as\ r\to\infty$, this proves that $\omega_0^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$ is a complete linear topological space.

Now we shall show that $\omega_.^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$ is a complete linear topological space. For this, since (x^r) is also a sequence in $\omega_.^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$ by definition of $\omega_.^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$ for each r there exists L^r with

$$\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{kl}^r+\Delta_t^v L^r}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]\to 0, as\ m,n\to\infty$$

Whence, from the fact that

$$\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}<\infty$$

From the definition of Modulus function, we have $M_{kl}\left(\left\|\frac{\Delta_t^v L^r-\Delta_t^v L^h}{\rho}\right\|\right)\to 0, as\ r,h\to\infty\ and\ so\ L^r\ converges\ to\ L$. Thus

$$\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}+L}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]\to 0$$

$as\ m,n\to\infty, x\in\omega_.^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$ and this complete the proof.

Theorem 2.3 Let $\mathcal{M}=(M_{kl})$ be a sequence of modulus functions, $A=\left(a_{m,n,k,l}\right)$ be a non-negative matrix such that $\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}<\infty$, then

(i). $\omega_.^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)\subset\omega_\infty^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$

(ii). $\omega_0^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)\subset\omega_\infty^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$

Proof. (i) Let $x\in\omega_.^{''}(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$. Then

$$\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$=\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}-L+L}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$\leq\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}-L}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$+(kl)^{-s}u_{k,l}[M_{k,l}\left(\left\|\frac{1}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}]\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}$$

Let there exist an integer m_l, such that $\left\|\frac{1}{\rho},z_1,\dots,z_{n-1}\right\|\leq m_l$. Thus, we have $\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]$

$$=\sum_{k,l=1,1}^{\infty,\infty}(kl)^{-s}u_{k,l}\left[a_{m,n,k,l}M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}-L}{\rho},z_1,\dots,z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$+m_l u_{k,l}M_{k,l}(1)\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}$$

Since $\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}<\infty$, and $x\in\omega_{.}''(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$

Thus, we have $x\in\omega_{\infty}''(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$.This complete the proof.

(ii). It is easy to prove in view of (i) so we omit the details.

Theorem 2.4 Let $\mathcal{M}=(M_{kl})$ be a sequence of modulus functions, $A=(a_{m,n,k,l})$ be a non-negative matrix such that $\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}<\infty$ and $\beta=\lim_{t\to\infty}\left(\frac{M_{kl}(t)}{t}\right)>0$, then

$$\omega_{.}''(\Delta_t^v,A,u,p,s,\|.,\dots,.\|)=\omega_{.}''(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$$

Proof. In order to prove that

$$\omega_{.}''(\Delta_t^v,A,u,p,s,\|.,\dots,.\|)=\omega_{.}''(\Delta_t^v,A,\mathcal{M},u,p,s,\|.,\dots,.\|)$$

It is sufficient to show that

$\omega_{.}^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|) \subset \omega_{.}^{''}(\Delta_t^v, A, u, p, s, \|., \dots, .\|)$

Now, let $\beta > 0$. By definition of β, we have $M_{kl}(t) \geq \beta(t), \forall\, t \geq 0.$ since $\beta > 0$, we have

$t \leq \frac{M_{kl}(t)}{\beta}, \forall t \geq 0.$

Let $x \in \omega_{.}^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$. Thus we have

$$\sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l}\left[a_{m,n,k,l} M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right]$$

$$\leq \frac{1}{\beta} \sum_{k,l=1,1}^{\infty,\infty} (kl)^{-s} u_{k,l}\left[a_{m,n,k,l} M_{k,l}\left(\left\|\frac{\Delta_t^v x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{kl}}\right]$$

Which implies that $x \in \omega_{.}^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$. This complete the proof.

Theorem 2.5 If $A = (a_{m,n,k,l})$ has only positive entries and $B = (b_{m,n,k,l})$ be a non-negative matrix such that $\left\{\frac{b_{m,n,k,l}}{a_{m,n,k,l}}\right\}$ is bounded then

$\omega_{\infty}^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|) \subset \omega_{\infty}^{''}(\Delta_t^v, B, \mathcal{M}, u, p, s, \|., \dots, .\|)$

Proof. It is easy to prove so we omit the details.

3. CONCLUSION

We discovered that the generalized spaces $\omega_0^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$, $\omega^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ and $\omega_{\infty}^{''}(\Delta_t^v, A, \mathcal{M}, u, p, s, \|., \dots, .\|)$ are not only linear over the field of complex numbers but also complete topological linear spaces. More over if there are two non negative four dimensional matrices such that under certain condition of boundedness one space will be contained in the other space.

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