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STEADY STATE SOLUTIONS FOR A FINITE CAPACITY QUEUEING SYSTEM WITH DETERMINISTIC SERVICE TIMES

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Abstract

In the paper an attempt has been made to compute some characteristics of M/D/1 queue with finite capacity. A closed form formulae for the distribution of the number of customers in the system has been derived. An explicit solution for mean queue length and average waiting time has also been provided, Embedded Markov Chain have been used for computing in this model.

Keywords: *Embedded Markov Chain. Generating function, Coefficient of Z-transform, Steady stat solutions, Mean queue length, average waiting time.*

1. Introduction:

In the present scenario, a number of servicing systems are providing services with constant servicing time. In the present paper, a model has been studied whose arrival pattern is in Poisson fashion and service facility is deterministic with finite capacity. M/D/1 queue is by far the simplest and the most general model. The model has a variety of applications not only in the telecommunication area, but also in the area of operation research, computer science and many other engineering areas. The model has been studied for a long time and is solved from a computational point of view.

Artalejo & Corral (1997) analyzed steady state solution of a single-server queue with linear repeated requests. Ahahiru sule Alfa and Gordan J. Fitzpatrick (1999) considered a Geo/D/1 queue operating under a hybrid FIFO/LIFO discipline and obtain the waiting time distribution. Bruneel & Kim (1993) gave a discrete time models for communication systems including ATM. Erlang (1909) studied a queueing system with Poisson arrival and deterministic i.e. constant, service time. This model is appropriate for continuous deterministic service time queueing systems, input can be seen as completely random or as a super position of the large number of processes. This follows from the fact that in situations with many sources each having a small generation rate.

Franx, G.J. (2001) gave a simple solution for the M/D/1 waiting time distribution". Gravey,Louvion et al. (1990) studied on the Geo/D/1 and Geo/D/1/N. Kumar, R and Sharma, S.K. (2012) studied M/M/1/N queueing system with relation of reneged customers. Such type of models was also studied by Roberts, Mocci and Virtamo (1996), Vicari and Trangia (1996), Brun and Garcia (2000) gave an analytical solution of finite capacity M/D/1 queues. Woodward (1994) studied communication and computer networks, modeling with discrete time queues.

2. Notations & Terminology:

λ : Arrival rate

T : Service time of each customer is the same and constant

ρ : Utilization factor = λT

$P_j(N)$: Probability of j-customer

$A_{j,l}$: Expected amounts of time that I-customer is present during a service time that is started with j-customers in the system.

$X_N(t)$: Number of customers in the system at time t

t_n : The date of the n^{th} customer departure

$q_j(t_n)$: Probability those j-customers are left behind by the n^{th} departure

α_l : probability of l arrivals during a customer service

A(Z) : Z-transform of the sequence

T_N : Average system time

W_N : Mean waiting time

$(b_n)_{n>o}$: Coefficients of the Z-transform B(Z)

3. The Model

In this section a model has been studied which has finite capacity with deterministic service time. In the queueing system there are N-1 places in the waiting room. The service mechanism is FCFS i.e. "First Come First Served" discipline. Some customers, who upon arrival see a full system are rejected, that is called lost customer.

Let $A_{j,I}$ the expected amount of time that I customers are present during a service time that is started with j-customers in the system. A scheme for the time average probabilities $P_{j(N)}$ is known for the model M/G/1/N, we get

$$P_l(N) = \lambda P_0(N) A_{1,l} + \lambda \sum_{j=1}^{l} p_j(N) A_{j,l} \quad 1 \le l \le N \qquad \ldots\ldots (1)$$

Since arrivals are Poisson distributed with rate λ and service time is deterministic i.e. constant T, we have

$$\alpha_l = \frac{\rho^l}{l!} e^{-\rho} \qquad \ldots\ldots (2)$$

Also probability transition matrix of the Imbedded Markov Chain, and the stationary distribution

$$\prod = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \ldots\ldots\ldots\ldots & \alpha_{N-2} & 1-\sum_{0}^{N-2}\alpha_1 \\ \alpha_0 & \alpha_1 & \alpha_2 \ldots\ldots\ldots\ldots & \alpha_{N-2} & 1-\sum_{0}^{N-2}\alpha_1 \\ 0 & \alpha_0 & \alpha_1 \ldots\ldots\ldots\ldots & \alpha_{N-3} & 1-\sum_{0}^{N-3}\alpha_1 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots & \alpha_0 & 1-\alpha_0 \end{bmatrix} \qquad \ldots.. (3)$$

$Q = [q_0, q_1 \ldots\ldots\ldots\ldots q_{N-1}]$, is a eigen vector of the matrix∏, so

$$Q\prod = Q \qquad \ldots\ldots (4)$$

Equation (3) implies that the stationary distribution Q verifies the following linear system

$$
\left.\begin{array}{l}
\alpha_0 q_0 + \alpha_0 q_1 = q_0 \\
\alpha_1 q_0 + \alpha_1 q_1 + \alpha_0 q_2 = q_1 \\
\alpha_2 q_0 + \alpha_2 q_1 + \alpha_1 q_2 + \alpha_0 q_3 = q_2 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\alpha_{N-2} q_0 + \alpha_{N-2} q_1 + \dots\dots\dots + \alpha_0 q_{N-1} = q_{N-2}
\end{array}\right\} \qquad \dots\dots (5)
$$

This is a linear system of N-1 equations involving the N-unknowns q_0, q_1, ----------------- q_{N-1}, $q_n = a_n q_0$, then

$a_0=1$, $a_1=e^{\rho}-1$ and that a_2 ....a_{N-1}

obey the following recursion :

$$a_n = e^{\rho}\left(a_{n-1} - \sum_{i=1}^{n-1} \alpha_i a_{n-i} - \alpha_{n-1} a_0 \right) \qquad \dots\dots (6)$$

Let A (z) be the z- transform

$$A(z) = \sum_{l=0}^{\infty} a_l z^l \qquad \dots\dots (7)$$

4. Generating Function A(Z)

Multiply by Z^n and summing over equation (6), we get

Let A(Z) be the Z-transform

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n a_0 z^n + \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} a_n a_i z^{n+i-1} \qquad \dots\dots (8)$$

Using equation (2) we get

$$A(z) = a_0 e^{-\rho} \sum_{n=0}^{\infty} \frac{(\rho z)^n}{n!} + \frac{1}{z} \sum_{i=1}^{\infty} a_i z^i \sum_{n=0}^{\infty} \alpha_n z^n = a_0 e^{-\rho} e^{\rho z} + \frac{1}{z}\left(\sum_{i=0}^{\infty} a_i z^i - a_0 \right) \sum_{n=0}^{\infty} \frac{(\rho z)^n}{n!} e^{-\rho}$$

$$A(z) = a_0 e^{\rho(z-1)} + \frac{e^{\rho(z+1)}}{z}(A(z) - a_0) \qquad = \frac{e^{\rho(z-1)}}{z}[z a_0 + A(z) - a_0]$$

$$A(z)\left(1 - z e^{\rho(1-z)}\right) = 1 - z$$

$$A(z) = \frac{1-z}{1 - z e^{\rho(1-z)}}$$

$$A(z) = (1-z)B(z) \qquad \dots\dots (9)$$

Where, B(z) $= \frac{1}{\left(1-ze^{\rho(1-z)}\right)}$

5. The Coefficients of Z-Transforms (b_n)

We have $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n - z\sum_{n=0}^{\infty} b_n z^n = b_0 + \sum_{n=1}^{\infty} b_n z^n - \sum_{n=1}^{\infty} b_{n-1} z^n$

$$\sum_{n=0}^{\infty} a_n z^n = b_0 + \sum_{n=0}^{\infty} (b_n - b_{n-1}) z^n \quad \text{...... (10)}$$

We can write $\sum_{n=0}^{\infty} a_n z^n = a_0 + \sum_{n=1}^{\infty} a_n z^n$ ...... (11)

By equation (10) and (11), we get

$$a_0 + \sum_{n=1}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) z^n \quad \text{...... (12)}$$

Which proves that $a_0 = 1$, $a_n = b_n - b_{n-1}$

By the z-transform F(z) as follows

$$F(z) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{(-1)^l}{l!} (n-l)^l e^{(n-l)\rho} \rho^l z^n \quad \text{...... (13)}$$

In equation (13), $n - l = m$, we have

$$F(z) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l}{l!} m^l e^{m\rho} \rho^l z^l z^m \quad \text{...... (14)}$$

Since $\sum_{m=0}^{\infty} \frac{(-1)^l}{l!} m^l \rho^l z^l = e^{-m\rho z}$

Adjusting this value in equation (14)

$$F(z) = \sum_{m=0}^{\infty} e^{m\rho(1-z)} z^m \quad \text{...... (15)}$$

If $\left|e^{\rho(1-z)} z\right| < 1$, then the series F(z) converges.

If $|z| < 1$, we have the following expression for the z-transform F (z)

$$F(z) = \frac{1}{1 - ze^{\rho(1-z)}} = B(z)$$

Hence we get $b_n = \sum_{l=0}^{n} \frac{(-1)^l}{l!}(n-l)^l e^{(n-l)\rho} \rho^l, \forall n \geq 1$ ...... (16)

By the Probability normalization

$$\sum_{l=0}^{N-1} q_1 = 1, \text{ Thus } q_0 = \frac{1}{\sum_{l=0}^{N-1} a_l} = \frac{1}{b_{N-1}} \quad \text{...... (17)}$$

and $q_n = a_n q_0$

$$q_n = \frac{b_n - b_{n-1}}{b_{N-1}}, \quad n=1, 2, \text{.......................} N-1 \quad \text{...... (18)}$$

6. Steady State Solution

The steady state probability distribution Q of the number of customers left behind by a departure differs from the queue length distribution.

$P = [P_0(N), \text{.....................................} P_N (N)]$, in M/D/1

queue with finite capacity.

We know that the probability of j-customer in case of M/G/1/N queues

$$q_j = \frac{P_j(N)}{1 - P_N(N)}, \quad \text{...... (19)}$$

j = 0, 1, 2 .... N-1

A simple conservation law holds that

$$\lambda(1 - P_N(N) = \frac{1}{T}(1 - P_0(N)) \quad \text{...... (20)}$$

using the result $\rho = \lambda T$

$$((1 - P_N(N)) = \frac{1}{\rho}(1 - P_0(N))$$

$$((1 - P_N(N)) = \frac{1}{\rho}(1 - (1 - P_N(N))q_0)$$

With the help of equation (17), we have,

$$1 - P_N(N) = \frac{1}{\rho}\left(1 - (1 - P_N(N))\frac{1}{b_{N-1}}\right)$$

Arranging above equation, we get

$$1 - P_N(N) = \frac{b_{N-1}}{1+\rho b_{N-1}} \qquad \text{...... (21)}$$

By equation (19), other probabilities are given by

$$P_0(N) = \frac{1}{1+\rho b_{N-1}} \qquad \text{...... (22)}$$

$$P_j(N) = \frac{b_j - b_{j-1}}{1+\rho b_{N-1}} \qquad \text{...... (23)}$$

Using equation (21) we have to calculate mean number of customers.

7. Mean Queue Length and Average Waiting Time

X_N be the mean number of customers of the M/D/1/N queue. Since, we know that

$$X_N = \sum_{l=0}^{N} l P_l(N) \qquad \text{...... (24)}$$

We have

$$X_N = N + \frac{\sum_{l=0}^{N-1} l(P_l - P_{l-1}) - N b_{N-1}}{1+\rho b_{N-1}} \qquad \text{...... (25)}$$

Solving above equation, we get mean numbers of customers

$$X_N = N - \frac{\sum_{l=0}^{N-1} b_l}{1+\rho b_{N-1}} \qquad \text{...... (26)}$$

Let T_N be the average system time, by Little's law

$$X_N = \lambda\,(1-P_N(N))\,T_N \qquad \text{......(27)}$$

By equation (27)

$$T_N = \frac{X_N}{\lambda\left(1-P_N(N)\right)} \qquad \text{...... (28)}$$

We use equation (21) & (24) in equation (28), for value of $1-P_N(N)$ and X_N.

$$T_N = \frac{1}{\lambda}\frac{1-\rho b_{N-1}}{b_{N-1}}\frac{N + N\rho b_{N-1} - \sum_{l=0}^{N-1} b_l}{1+\rho b_{N-1}}$$

$$T_N = T\left(N - \frac{\sum_{l=0}^{N-1} b_l - N}{\rho b_{N-1}} \right) \qquad \text{...... (29)}$$

We know that,

Waiting time = Average system time – Service time

i.e. $W_N = T_N - T = T\left(N - \frac{\sum_{l=0}^{N-1}(b_l - N)}{\rho b_{N-1}} \right) - T$

Average waiting time

$$W_N = \left(N - 1 - \frac{\sum_{l=0}^{N-1} b_l - N}{\rho b_{N-1}} \right) T \qquad \text{...... (30)}$$

8. Conclusion:

In the preceding sections, the expressions for mean queue length and average waiting time have been obtained. A closed form formula for the distribution of the number of customers in the M/D/1 queue capacity has also been provided.

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