International Research Journal of Mathematics, Engineering and IT
Vol. 3, Issue 10, October 2016 IF- 3.563 ISSN: (2349-0322)

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MATRIX TRANSFORMATIONS BETWEEN GENERALIZED RIESZ SEQUENCE SPACES OF NON ABSOLUTE TYPE

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ABSTRACT

The main purpose of this paper is to characterize the classes $(r^q(u,p,s), bs), (r^q(u,p,s), cs)$ and $(r^q(u,p,s), c_0s)$ of the infinite matrices, where bs, cs and c_0s denote the space of all bounded series, the space of all convergent series and the space of series converging to zero, respectively.

KEYWORDS: Generalized Riesz Sequence Space, Matrix Transformations, Paranormed Sequence Spaces, Sequence Space of Non Absolute Type,.

Mathematical Subject Classification (AMS 2010): 46A45, 46B25, 40C05

1. INTRODUCTION

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of non negative integers, the set of real numbers and the set of complex numbers, respectively. Let ω denote the space of all sequences (real or complex), ℓ_∞ *and* c, respectively denote the space of all bounded sequences and the space of all convergent sequences. A linear topological space X over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a sub additive function $h : X \to \mathbb{R}$ such that $h(\theta) = 0, h(-x) = h(x)$ and scalar multiplication is continuous, that is , $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$, as $n \to \infty$ for all $\alpha' s$ in $\mathbb{R}$ and $x' s$ in X, where θ is the zero vector in the linear space X. Assuming here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ *and* $M = \max(1, H)$. Then the linear space $\ell(p)$ *and* $\ell_\infty(p)$ were defined by Maddox [1], (see also [2], [3] and [4]) as follows:

$$\ell(p) = \{ x = (x_k) : \sum_k |x_k|^{p_k} < \infty \} \text{ with } 0 < p_k \leq H < \infty.$$
$$\ell_\infty(p) = \left\{ x = (x_k) : \sup_k |x_k|^{p_k} < \infty \right\}$$

which are complete spaces paranormed, respectively by

$$g_1(x) = [\sum_k |x_k|^{p_k}]^{\frac{1}{M}} \quad \text{and} \quad g_2(x) = \sup_k |x_k|^{p_k/M} \text{ iff } \inf_k p_k > 0.$$

We shall assume throughout that $p_k^{-1} + t_k^{-1} = 1$ and provided $1 < inf p_k \leq H < \infty$.

In [5] Stieglitz and Tietz defined

$$cs = \{ x = (x_k) : (\sum_{k=1}^{n} x_k) \in c \} \tag{1.1}$$
$$c_0 s = \{ x = (x_k) : (\sum_{k=1}^{n} x_k) \in c_0 \} \tag{1.2}$$
$$bs = \{ x = (x_k) : (\sum_{k=1}^{n} x_k) \in \ell_\infty \} \tag{1.3}$$

For the sequence spaces X and Y, define the set

$$M(X, Y) = \{ z = (z_k) \in \omega : xz = (x_k z_k) \in Y, \text{for all } x \in X \}. \tag{1.4}$$

With the notion of (1.4) the $\alpha-, \beta - and\, \gamma -$ duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ and are defined by

$$X^\alpha = M(X, \ell_1), X^\beta = M(X, cs) \text{ } and \text{ } X^\gamma = M(X, bs).$$

If a sequence space X paranormed by h contains a sequence (b_n) with the property that for every $x \in X$, there is a unique sequence of scalars (α_n) such that

$$\lim_n h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0.$$

Then (b_n) is called a Schauder basis or (briefly basis) for X. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), written as $x = \sum_k \alpha_k b_k$.

Let X and Y be a two subsets of ω. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then the matrix A defined the A-transformation from X into Y, if for every sequence $x = (x_k) \in X$. The sequence $Ax = (Ax)_n$, the A-transform of x exists and is in Y, where $= (Ax)_n = \sum_k a_{nk} x_k$. For simplicity of notation, here and what follows, the summation without limits runs from 0 to ∞. By (X, Y), we denote the class of all matrices. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{ x = (x_k) : (Ax) \in X \}. \tag{1.5}$$

2. SOME BASIC DEFINITIONS AND LEMMAS

In this section we give some important definitions which shall be used in this work.

Definition 2.1 Let $q = (q_i)$ be a sequence of positive real numbers and let write

$Q_n = \sum_i^n q_i$ for $n \in \mathbb{N}$. Then the matrix $R^q = (r_{nk}^q)$ of the Riesz mean (R, q_n) is given by

$$\left(r_{nk}^q\right) = \begin{cases} \frac{\mathrm{q_k}}{\mathrm{Q_k}} & \text{if } 0 \leq \mathrm{k} \leq \mathrm{n} \\ 0, & \text{if } \mathrm{k} > n \end{cases} \tag{2.1}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty, as\ n \to \infty$.

(See Petersen [6] and [7])

Recently, the following Riesz sequence space was introduced.

Definition 2.2 (Sheik and Ganie [8]) defined and studied the Riesz sequence space $r^q(u.p)$ of non –absolute type by

$r^q(u.p) = \{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} | \frac{1}{Q_n} \sum_{k=1}^{n} u_k q_k x_k |^{p_k} < \infty \}$, where $0 < p_k \leq H < \infty$.

Definition 2.3 (Fazlur Rahman and Rezaul Karim [9]) For $s \geq 0$, we define

$r^q(u, p, s) = \{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} | \frac{1}{Q_n^{s+1}} \sum_{k=1}^{n} u_k q_k x_k |^{p_k} < \infty \}$,

If s = 0, then $r^q(u, p, s)$ reduces to $r^q(u.p)$ which is defined and studied in [7].

Now the sequence $y = (y_k)$ is defined by

$$y_k = \frac{1}{Q_k^{s+1}} \sum_{j=1}^{k} u_j q_j x_j \tag{2.2}$$

Note the following inequality (see [10]), which will be used in this paper.

For any integer E > 1 and any two complex numbers $a\ and\ b$ we have

$$|a.b| \leq E(|a|^t E^{-t} + |b|^p) \tag{2.3}$$

Lemma 2.1 ([9], Theorem 1.1)

The Riesz sequence space $r^q(u.p, s)$ is a complete linear metric space paranormed by

$$g(x) = (\sum_{n=1}^{\infty} | \frac{1}{Q_n^{s+1}} \sum_{k=1}^{n} u_k q_k x_k |^{p_k})^{1/M}.$$

Lemma 2.2 ([9], Theorem 2.1). Let $1 < p_k \leq H < \infty$, for every $k \in \mathbb{N}$.

Define the sets $D_1(u, p, s)$and $D_2(u, p, s)$ as follows:

$$D_1(u,p,s) = \bigcup_{E>1}\{ a=(a_k) \in \omega\colon \sup_{n\in\mathcal{F}} \sum_k | \sum_{n\in\mathbb{N}} (-1)^{n-k} \frac{a_n}{u_n q_n} Q_k^{s+1} E^{-1}|^{t_k} < \infty \} \quad (2.4)$$

$$\left. \begin{array}{c} D_2(u,p,s) = \bigcap_{E>1}\{ a=(a_k) \in \omega\colon \sum_k |\Delta(\frac{a_k}{u_k k}) Q_k^{s+1} E^{-1}|^{t_k} < \infty \\ \text{and } ((\frac{a_k}{u_k q_k} Q_k^{s+1} E^{-1})^{t_k} \in \ell_\infty \} \end{array} \right\} \quad (2.5)$$

Then,

$$[r^q(u,p,s)]^\alpha = D_1(u,p,s) \text{ and } [r^q(u,p,s)]^\beta = [r^q(u,p,s)]^\gamma = D_2(u,p,s)$$

Lemma 2.3 ([9], Theorem 2.2): Let $0 < p_k \leq 1$. for every $k \in \mathbb{N}$. Define $D_3(u,p,s)$ and $D_4(u,p,s)$ as

$$D_3(u,p,s) = \{ a=(a_k) \in \omega\colon \sup_{n\in\mathcal{F}} \sup_k | \sum_{n\in\mathbb{N}} (-1)^{n-k} \frac{a_n}{u_n q_n} Q_k^{s+1} E^{-1}|^{p_k} < \infty \} \quad (2.6)$$

$$D_4(u,p,s) = \left. \begin{array}{c} a=(a_k) \in \omega\colon \sup_k |\Delta \frac{a_n}{u_n q_n} Q_k^{s+1} E^{-1}|^{p_k} < \infty \\ and \ \sup_k | \frac{a_n}{u_n q_n} Q_k^{s+1} E^{-1}|^{p_k} < \infty \end{array} \right\} \quad (2.7)$$

Then,

$$[r^q(u,p,s)]^\alpha = D_3(u,p,s) \ and \ [r^q(u,p,s)]^\beta = [r^q(u,p,s)]^\gamma = D_4(u,p,s).$$

Lemma 2.4 ([9], Theorem 3.1)

(i) Let $1 < p_k \leq H < \infty$, for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), \ell_\infty)$ if and only if there exists an integer E > 1 such that

$$U(E) = \sup_n \sum_k | \Delta \frac{a_{nk}}{u_k q_k} Q_k^{s+1} E^{-1}|^{t_k} < \infty \quad (2.8)$$

and

$$(\frac{a_{nk}}{u_k q_k} Q_k^{s+1} E^{-1})^{t_k} \in \ell_\infty \quad (2.9)$$

(ii) Let $0 < p_k \leq 1$, for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), \ell_\infty)$ if and only if

$$\sup_n |\Delta \frac{a_{nk}}{u_k q_k} Q_k^{s+1} E^{-1}|^{p_k} < \infty \quad (2.10)$$

Lemma 2.5 ([9], Theorem 3.2). Let $1 < p_k \leq H < \infty$, for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), c)$ if and only if conditions (2.8) and (2.9) hold, and there is a sequence of scalars (α_k) such that

$$\lim_n \Delta \frac{a_n}{u_n q_n} Q_k^{s+1} = \beta_k \quad (2.11)$$

Lemma 2.6 ([9], Corollary 3.1). Let $1 < p_k \leq H < \infty$, for each $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), c_0)$ if and only if the conditions (2.8), (2.9) hold and (2.11) holds with $\beta_k = 0$, for each $k \in \mathbb{N}$.

3. MAIN RESULTS

In this section we characterize the matrix classes $(r^q(u,p,s), bs), r^q(u,p,s), cs)$, and $r^q(u,p,s), c_0 s)$. We shall prove the following results.

Theorem 3.1 (i) Let $1 < p_k \leq H < \infty$, for every $k \in \mathbb{N}$. Then$A \in (r^q(u,p,s), bs\ )$ if and only if there exists an integer $B > 1$ such that

$$\sup_n \sum_k |\, \Delta \frac{a_{nk}}{u_k q_k} Q_k^{s+1} B^{-1} |^{t_k} < \infty, n \in \mathbb{N}. \tag{3.1}$$

and

$$\sup_k | \frac{a_{nk}}{u_k q_k} Q_k^{s+1} B^{-1} |^{t_k} < \infty\ , n \in \mathbb{N}. \tag{3.2}$$

(ii) Let $0 < p_k \leq 1$, for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), bs)$ if and only if

$$\sup_n |\Delta \frac{a_{nk}}{u_k q_k} Q_k^{s+1} |^{p_k} < \infty \tag{3.3}$$

Proof. Let us define the matrix $E = (e_{nk})$ by $e_{nk} = a(n,k)$ for all $n \in \mathbb{N}$, consider now equality derived from the $n; mth$ partial sum of the series $\sum_j^n \sum_k^m a_{jk}\ x_k$ as $m \to \infty$,

$$\sum_j^n \sum_k a_{jk}\ x_k = \sum_k e_{nk}\ x_k \text{ for all } n,k \in \mathbb{N}.$$

Therefore , bearing in mind the fact that the space $bs \text{ and } \ell_\infty$ are linearly isomorphic, one can easily see that $Ax \in bs$ whenever $x \in r^q(u,p,s)$ if and only if $Ex \in \ell_\infty$, whenever $x \in r^q(u,p,s)$.

Moreover, let $A \in (r^q(u,p,s), bs)$. Then $A_n(x) = \sum_k a_{nk}\ x_k$ exists for $x \in r^q(u,p,s)$ and this implies that $(a_{nk}) \in [r^q(u,p,s)]^\beta$ for every $n,k \in \mathbb{N}$. So by lemma 2.1, the necessities of (3.1) and (3.2) hold.

Sufficiency. Suppose the conditions (3.1) and (3.2) hold.

For $m,n \in \mathbb{N}$, consider the equation

$$\sum_{k=1}^m a_{nk} x_k = \sum_{k=1}^{m-1} \Delta \left(\frac{a_{nk}}{u_k q_k}\right) Q_k^{s+1} y_k + \frac{a_{nm}}{u_m q_m} Q_m^{s+1} y_m \tag{3.3}$$

when $m \to \infty$, then from (3.3) we have

$$\sum_{k=1}^\infty a_{nk} x_k = \sum_{k=1}^\infty \Delta \left(\frac{a_{nk}}{u_k q_k}\right) Q_k^{s+1} y_k \tag{3.4}$$

Using inequality (2.3) and (3.4)

$$\sup_n |\sum_{k=1}^{\infty} a_{nk}\, x_k| \le \sup_n \sum_{k=1}^{\infty} |\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k^{s+1}||y_k|$$

$$\le B \sup_n [\sum_{k=1}^{\infty} |\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k^{s+1}|^{t_k} B^{-t_k} + \sum_k |y_k|^{p_k}]$$

$$\le B[U(B) + g_1^M(y) < \infty$$

This shows that $A \in (r^q(u,p,s), bs)$.

(ii) The proof of the second part is similar to that of the part (i). Therefore it is omitted.

Theorem 3.2: Let $1 < p_k \le H < \infty$, for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), cs)$ if and only if (3.1), (3.2) hold and there is a sequence of scalars (α_k) such that

$$\lim_n \Delta \frac{a_n - \alpha_k}{u_k q_k} Q_k^{s+1} = \beta_k \text{ for every } n.k \in \mathbb{N}. \quad (3.5)$$

Proof. Necessity Suppose $A \in (r^q(u,p,s), cs)$ and $1 < p_k \le H < \infty$, Then the A-transformation of $r^q(u,p,s)$ exists and belong to cs. Hence, $(a_{nk}) \in [r^q(u,p,s)]^\beta$. By Lemma (2.1), the necessities of (3.1) and (3.2) hold. For the necessity of condition (3.5), we take for each fixed k, a sequence $x^{(k)} = \left(x_n^{(k)}(q)\right)\ in\ r^q(u,p,s)$ with

$$x_n^{(k)}(q) = \begin{cases} (-)^{n-k} \frac{Q_k^{s+1}}{u_n q_n}, & \text{if } k \le n \le k+1 \\ 0\,, & \text{if } 0 \le n < k \text{ or } n > k+1 \end{cases}$$

Then for each $k \in \mathbb{N}$, we have $Ax^k \in cs$, which shows that

$(\Delta \frac{a_n - \alpha_k}{u_k q_k} Q_k^{s+1}) \in c$. This proves the necessity of the condition (3.5).

Sufficiency. Suppose that the conditions (3.1), (3.2) and (3.5) hold. Then for $x \in r^q(u,p,s)$, we have $(a_{nk}) \in [r^q(u,p,s)]^\beta$ for each n and so $Ax = \sum_k a_{nk}\, x_k$ exists.

For every $m, n \in \mathbb{N}$, we have

$$\sum_{k=1}^{m} |\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k^{s+1} B^{-1}|^{p_k} \le \sup_n \sum_k |\,\Delta \frac{a_{nk}}{u_k q_k} Q_k^{s+1} B^{-1}|^{P_k}$$

Letting $m, n \to \infty$, together with (3.1) and (3.5) give

$$.\sum_k |\,\Delta \frac{\alpha_k}{u_k q_k} Q_k^{s+1} B^{-1}|^{P_k} < \infty \quad (3.6)$$

Also by letting $n \to \infty$, we have from (3.2) that

$(\frac{a_{nk}}{u_k q_k} Q_k^{s+1} B^{-1})^{p_k} \in \ell_\infty$, which leads together with (3.6) that

$(\alpha_k) \in D_2(u,p,s)$. Thus the series $\sum_k \alpha_k x_k$ converges for every $x \in r^q(u,p,s)$.

Writing $a_{nk} - \alpha_k$ for all a_{nk} we have from (3.4)

$$\sum_{k=1}^{\infty}(a_{nk} - \alpha_k)x_k = \sum_{k=1}^{\infty} \Delta\left(\frac{a_{nk} - \alpha_k}{u_k q_k}\right) Q_k^{s+1} y_k \tag{3.7}$$

Comparing this with lemma (3.4) with $\beta_k = 0$, for all $k \in \mathbb{N}$.

We get the matrix $(\Delta\left(\frac{a_{nk} - \alpha_k}{u_k q_k}\right) Q_k^{s+1})_{n,k\in\mathbb{N}} \in (\ell(p), c_0)$

Thus, by (3.7) we get

$$\lim_n \sum_k (a_{nk} - \alpha_k)x_k = 0 \tag{3.8}$$

Now, by combining (3.8) with the above results on can see that $Ax \in cs$. Hence the proof.

Theorem 3.3 Let $1 < p_k \leq H < \infty$, for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p,s), c_0 s)$ if and only if conditions (3.1), (3.2) hold and (3.5) also holds with $\beta_k = 0$ for each $k \in \mathbb{N}$.

Proof. This may be proved using similar argument as in the above theorem (Theorem 3.2) and therefore omitted.

4. CONCLUSION

Recently, several authors defined and studied Riesz sequence space $r^q(u,p)$ of non absolute type. Furthermore, many generalizations of the above sequence space were introduced such as $r^q(u,p,s)$ and characterization of the classes $(r^q(u,p,s), \ell_\infty)$, $(r^q(u,p,s), c)$ and $(r^q(u,p,s), c_0)$ were equally obtained by (Fazlur Rahman et al [9]). In this paper , we characterized the classes of the infinite matrices $(r^q(u,p,s), bs)$, $(r^q(u,p,s), cs)$ and $(r^q(u,p,s), c_0 s)$ as our main results. There is room for more characterizations.

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