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Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

SUCCESSIVE OCCURRENCE OF DIGIT 0 IN ALL BASE *b* NATURAL NUMBERS LESS THAN b^n

Neeraj Anant Pande

Associate Professor, Department of Mathematics and Statistics,
Yeshwant Mahavidyalaya (College), Nanded – 431602, Maharashtra, India

ABSTRACT

General base b positive integers from 1 to one less than b^n, for any positive integer n, are under consideration. 0 as significant digit is analyzed for its presence within natural numbers in the range, the count of its single and multiple occurrences and the first and last numbers for its appearances, formulae are derived for each of these. Examples are illustrated with more preferred particular base 16.

Keywords: Successive occurrences, digit 0, integers in general base.

Mathematics Subject Classification 2010: 11Y35, 11Y60, 11Y99.

1. Introduction

Positive integers

$$1, 2, 3, \cdots$$

can be represented in any natural number base $b > 1$. The decimal base 10 is in routine use, but we know that other bases are equally important, like for machine level calculations of electronic computers, base $b = 2$ is in execution, for advanced level calculations on the same devices, hexadecimal base $b = 16$ is in use.

Here, we have used the term number to mean the natural number. All base b numbers m in the range, with $1 \le m < b^n$ are extensively analyzed. Their one common property is that all of them have n or fewer significant digits in base b expansion.

All types of occurrences of all non-zero digits are treated in usual decimal numbers in [2], [3], [4] in general base numbers in [8], [9], [10].Similar treatment for presence of 0 is found in [5], [6], [7]. Regardless of the base b, digit 0 is inevitable in representation of numbers in a system with place value property [1]. Significant 0's, i.e., 0's after some non-zero digit, in succession are considered in this analysis of numbers in aforementioned range.

2. SUCCESSIVE OCCURRENCE OF DIGIT 0

All occurrences of 0's in all base b numbers less than b^n, have already been analyzed in [11].

Theorem 1 : If r, n and b are positive integers with $1 < r<n$ and $b > 1$, then the number of base b numbers containing exactly r number of significant digit 0's in the range $1 \le m < b^n$ is

$$ {}_{0}^{A}\underset{b}{O}{}_{r}^{n} = \sum_{j=r+1}^{n} {}^{j-1}C_r (b-1)^{j-r} \ . $$

where the notation ${}_{0}^{A}\underset{b}{O}{}_{r}^{n}$ is for number of base b numbers less than b^n with r number of 0's.

No we focus on successive occurrences of 0's in all base b numbers less than b^n.

For initial inspection, specific base $b = 16$ is chosen.

As mentioned earlier, choosing consecutive ranges $1 \le m < 16^n$, for various positive integers n, we have determined the count of successive occurrence of 1 and 2 0's in hexadecimal numbers within ranges, by using Java program till 16^{15}, which happens to be little more than a quintillion, i.e., 10^{18}. Owing to our familiarity with decimal usage, these figures are mentioned in usual base 10.

Table 1: Number of Hexadecimal Numbers in Various Ranges with Single and Double Successive 0's in Their Digits

Sr. No.	Numbers Range <	Number of Hexadecimal Numbers with Single (Successive) 0	Number of Hexadecimal Numbers with 2 Successive 0's
1.	16^1	0	0
2.	16^2	15	0
3.	16^3	465	15
4.	16^4	10,590	465
5.	16^5	213,090	10,590
6.	16^6	4,009,965	213,090
7.	16^7	72,353,715	4,009,965
8.	16^8	1,268,369,340	72,353,715
9.	16^9	21,771,494,340	1,268,369,340
10.	16^{10}	367,761,728,715	21,771,494,340
11.	16^{11}	6,134,265,634,965	367,761,728,715
12.	16^{12}	101,281,580,088,090	6,134,265,634,965
13.	16^{13}	1,658,237,634,775,590	101,281,580,088,090
14.	16^{14}	26,958,773,523,447,465	1,658,237,634,775,590
15.	16^{15}	435,659,737,878,916,201	26,958,773,523,447,465

The first range $1 \le m < 16^1 = 16$, doesn't have any 0 at all. There are only 15 numbers in it which are all fifteen digits 1, 2, ⋯, F and 0 is not there.

In the second range $1 \leq m < 16^2 = 256$, single 0 occurs 15 times. Its instances, as specified in [11], are in numbers

10, 20, 30, 40, 50, 60, 70, 80, 90, A0, B0, C0, D0, E0, and F0.

all at unit's places. So, first occurrence is in initial block and it is ${}^{2\text{-}1}C_1 15^{2\text{-}1} = 15$ times. Single occurrence is always qualified to be successive due to absence of non-successive character.

In the third range, $1 \leq m < 16^3 = 4{,}096$, as stated in [4], single 0 is seen 465 times. It occurs in earlier 15 numbers

10, 20, 30, · · · , F0,

and then additionally in

110, 120, 130, · · · , 1F0, 210, 220, 230, · · · , 2F0, ⋯ , F10, F20, F30, · · · , FF0,

101, 102, 103, · · · , 10F, 201, 202, 203, · · · , 20F, ⋯ , F01, F02, F03, · · · , F0F,

which is ${}^{3\text{-}1}C_1 15^{3\text{-}1} = 2 \times 15^2 = 450$ times; totaling to 15 + 450 = 465 times. As earlier, since single 0 can be considered as successive, these happen to fall in the category of successive occurrences.

In this range, double zeros occur in numbers

100, 200, 300, · · · , F00,

with count is ${}^{3\text{-}1}C_2 15^1 = 15$. All of them are already successive!

In the fourth range, $1 \leq m < 16^4 = 65{,}536$, as determined in [11], single 0 appears in earlier 465 numbers

10, 20, 30, · · · , F0,

110, 120, 130, · · · , 1F0, 210, 220, 230, · · · , 2F0, ⋯ , F10, F20, F30, · · · , FF0.

101, 102, 103, · · · , 10F, 201, 202, 203, · · · , 20F, ⋯ , F01, F02, F03, · · · , F0F,

and then additionally in

1110, 1120, · · · , 11F0, 1210, 1220, · · · , 12F0, ⋯ , 1F10, 1F20, · · · , 1FF0,

2110, 2120, · · · , 21F0, 2210, 2220, · · · , 22F0, ⋯ , 2F10, 2F20, · · · , 2FF0,

⋮

F110, F120, · · · , F1F0, F210, F220, · · · , F2F0, ⋯ , FF10, FF20, · · · , FFF0,

1101, 1102, · · · , 110F, 1201, 1202, · · · , 120F, ⋯ , 1F01, 1F02, · · · , 1F0F,

2101, 2102, · · · , 210F, 2201, 2202, · · · , 220F, ⋯ , 2F01, 2F02, · · · , 2F0F,

$$\vdots$$

$$\text{F101, F102, } \cdots \text{, F10F, F201, F202, } \cdots \text{, F20F, } \cdots \text{, FF01, FF02, } \cdots \text{, FF0F,}$$

$$\text{1011, 1012, } \cdots \text{, 101F, 1021, 1022, } \cdots \text{, 102F, } \cdots \text{, 10F1, 10F2, } \cdots \text{, 10FF,}$$

$$\text{2011, 2012, } \cdots \text{, 201F, 2021, 2022, } \cdots \text{, 202F, } \cdots \text{, 20F1, 20F2, } \cdots \text{, 20FF,}$$

$$\vdots$$

$$\text{F011, F012, } \cdots \text{, F01F, F021, F022, } \cdots \text{, F02F, } \cdots \text{, F0F1, F0F2, } \cdots \text{, F0FF,}$$

They give count ${}^{4\text{-}1}C_1\ 15^3 = 3 \times 15^3 = 10{,}125$, totaling to 465 + 10,125 = 10,590. They too are single and hence are considered successive.

Now in this range double 0 occurs in earlier 15 numbers

$$\text{100, 200, 300, } \cdots \text{, F00,}$$

and then additionally they occur successively in

$$\text{1100, 1200, } \cdots \text{, 1F00, 2100, 2200, } \cdots \text{, 2F00, } \cdots \text{, F100, F200, } \cdots \text{, FF00,}$$

$$\text{1001, 1002, } \cdots \text{, 100F, 2001, 2002, } \cdots \text{, 200F, } \cdots \text{, F001, F002, } \cdots \text{, F00F.}$$

They count ${}^{3\text{-}1}C_1 15^2 = 2 \times 15^2 = 450$, totaling to 15 + 450 = 465.

Thus all numbers in above table are explainable.

Taking hint from [6], we have formulated the count of occurrences of successive multiple 0's in our ranges.

Notation : We put forth new notation $\underset{b}{{}_0^S O_r^n}$ for number of base b numbers less than b^n with r number of successive 0's.

Theorem 2 : If r, n and b are positive integers with $1 < r < n$ and $b > 1$, then the number of base b numbers containing exactly r number of significant successive digit 0's in the range $1 \le m < b^n$ is

$$\underset{b}{{}_0^S O_r^n} = \sum_{j=2}^{n-(r-1)} {}^{j-1}C_1 (b-1)^{j-1} .$$

Proof 1. Let n, r and b be positive integers with $1 < r < n$ and $b > 1$. Base b will have b number of digits. If $b \le 10$, these digits are preferably denoted by 0, 1, 2, $\cdots$,$b-1$. In case $b > 10$, to denote first 10 digits we adopt usual symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and remaining successive digits are represented by A, B, C, $\cdots$,X where X is not essentially alphabet X but just a symbol to designate that alphabet at position equal to 9 more than its own position in the list of alphabets.

We prove the result by induction on both n and r. The cases of initial values of r are already discussed and validity of the formula is verified above. For any positive r, the minimum value of n is $r + 1$. In this case, there is only one possible position for all consecutive 0's, viz., last succession, and then there are exactly $(b – 1)$ occurrences of r successive 0's in

$$1\underbrace{000\cdots0}_{r\text{ times}},\ 2\underbrace{000\cdots0}_{r\text{ times}},\cdots,\ X\underbrace{000\cdots0}_{r\text{ times}}.$$

Also as per formula, this count is

$$ {}_{\substack{0\\b}}^{S}O_r^{n=r+1} = \sum_{j=2}^{r+1-(r-1)} {}^{j-1}C_1(b-1)^{j-1} = {}^{1}C_1(b-1)^{2-1} = (b-1)$$

So, the formula hold good in this case. The count of successive occurrences begins with $(b – 1)$.

The next value of n is $r + 2$. Here except the forefront position, there are $n – 1$, in this case 2, possible positions that can be occupied by r number of successive 0's, viz., last and middle.

The actually mention these gives :

$$1\underbrace{000\cdots0}_{r\text{ times}},\ 2\underbrace{000\cdots0}_{r\text{ times}},\cdots,\ X\underbrace{000\cdots0}_{r\text{ times}}, \qquad (= b-1)$$

$$11\underbrace{000\cdots0}_{r\text{ times}},\cdots,1X\underbrace{000\cdots0}_{r\text{ times}},\ 21\underbrace{000\cdots0}_{r\text{ times}},\cdots,2X\underbrace{000\cdots0}_{r\text{ times}},\cdots,X1\underbrace{000\cdots0}_{r\text{ times}},\cdots,XX\underbrace{000\cdots0}_{r\text{ times}}$$

$$(= (b-1)\times(b-1))$$

$$1\underbrace{000\cdots0}_{r\text{ times}}1,\cdots,1\underbrace{000\cdots0}_{r\text{ times}}X,2\underbrace{000\cdots0}_{r\text{ times}}1,\cdots,2\underbrace{000\cdots0}_{r\text{ times}}X,\cdots,X\underbrace{000\cdots0}_{r\text{ times}}1,\cdots,X\underbrace{000\cdots0}_{r\text{ times}}X$$

$$(= (b-1)\times(b-1))$$

In addition to earlier $(b – 1)$ occurrences of r successive 0's in the first row, now there are twice $(b – 1)^2$ additional such occurrences, so total occurrences of r successive 0's is given here by

Previous + ${}^{2}C_1(b – 1)^2$ and hence equals

$$\begin{aligned}(b-1) + {}^{2}C_1(b-1)^2 &= {}^{1}C_1(b-1)^{r+1-r} + {}^{r+2-(r-1)-1}C_1(b-1)^{r+2-(r-1)-1}\\ &= \sum_{j=2}^{r+2-(r-1)} {}^{j-1}C_1(b-1)^{j-1}\\ &= {}_{\substack{0\\b}}^{S}O_r^{n=r+2}\end{aligned}$$

asserting the formula is true in this case.

Continuing this indefinitely, the formula is proved for all positive integers r and $n > r$.

We provide an alternative proof for this result.

Proof 2. We will have an easy alternative proof of this Theorem coming as an application of Theorem in [11].

Our range is $1 \le m < b^n$. We want occurrences of r successive 0's in it. Because we want all r number of 0's to occur one after the other, we can consider that instead of r digits, they together form only one unit block '$\underbrace{000\cdots0}_{r\text{ digits}}$'. Then there remain only $n-(r-1)$ digit units. Changing n to $n-(r-1)$ and r to 1 in right hand side of the formula and A for all in notation by S for successive, we get the formula for occurrences of r successive 0's as

$$ {}^{S}_{0}\underset{b}{O}{}^{n}_{r} = \sum_{j=1+1}^{n-(r-1)} {}^{j-1}C_1 (b-1)^{j-1} = \sum_{j=2}^{n-(r-1)} {}^{j-1}C_1 (b-1)^{j-1} . $$

The first table above is now extendable to higher occurrences of successive 0's in hexadecimal numbers by employing this formula.

Table 2: Number of Hexadecimal Numbers in Various Ranges with MultipleSuccessive 0's in Their Digits

Sr. No.	Number Range <	Number of Hexadecimal Numbers with		
		3 Successive 0's	4 Successive 0's	5 Successive 0's
1.	16^4	15	0	0
2.	16^5	465	15	0
3.	16^6	10,590	465	15
4.	16^7	213,090	10,590	465
5.	16^8	4,009,965	213,090	10,590
6.	16^9	72,353,715	4,009,965	213,090
7.	16^{10}	1,268,369,340	72,353,715	4,009,965
8.	16^{11}	21,771,494,340	1,268,369,340	72,353,715
9.	16^{12}	367,761,728,715	21,771,494,340	1,268,369,340
10.	16^{13}	6,134,265,634,965	367,761,728,715	21,771,494,340
11.	16^{14}	101,281,580,088,090	6,134,265,634,965	367,761,728,715
12.	16^{15}	1,658,237,634,775,590	101,281,580,088,090	6,134,265,634,965

Table 2: Continued …

Sr. No.	Number Range <	Number of Hexadecimal Numbers with		
		6 Successive 0's	7 Successive 0's	8 Successive 0's
1.	16^7	15	0	0
2.	16^8	465	15	0
3.	16^9	10,590	465	15
4.	16^{10}	213,090	10,590	465
5.	16^{11}	4,009,965	213,090	10,590
6.	16^{12}	72,353,715	4,009,965	213,090
7.	16^{13}	1,268,369,340	72,353,715	4,009,965
8.	16^{14}	21,771,494,340	1,268,369,340	72,353,715
9.	16^{15}	367,761,728,715	21,771,494,340	1,268,369,340

Table 2: Continued …

Sr. No.	Number Range <	Number of Hexadecimal Numbers with		
		9 Successive 0's	10 Successive 0's	11 Successive 0's
1.	16^{10}	15	0	0
2.	16^{11}	465	15	0
3.	16^{12}	10,590	465	15
4.	16^{13}	213,090	10,590	465
5.	16^{14}	4,009,965	213,090	10,590
6.	16^{15}	72,353,715	4,009,965	213,090

Table 2: Continued …

Sr. No.	Number Range <	Number of Hexadecimal Numbers with		
		12 Successive 0's	13 Successive 0's	14 Successive 0's
1.	16^{13}	15	0	0
2.	16^{14}	465	15	0
3.	16^{15}	10,590	465	15

3. FIRST OCCURRENCE OF SUCCESSIVE DIGIT 0'S

In the first number in appropriate ranges containing required number of 0's, the 0's happen to be already successive and hence the general formula for first number containing specific number of 0's also applies the one containing successive number of 0's.

Formula 1 : If n, r and b are natural numbers with $b > 1$, then the first occurrence of r number of successive 0's in base b numbers in range $1 \le m < b^n$ is

$$f = \begin{cases} - \ , \text{if } r \ge n \\ b^r \ , \text{if } r < n \end{cases}.$$

4. LAST OCCURRENCE OF SUCCESSIVE DIGIT 0'S

The case of the last number in range containing required number of successive 0's is not different from general, like above.

Table 3: Last Hexadecimal Numbers with Multiple Successive 0's in their Digits in Various Base Power Ranges

Sr. No.	Number Range < → Last Hexadecimal Number with Successive↓	16^1	16^2	16^3	16^4	16^5	16^6	16^7	16^8	16^9
1.	1 0	-	F0	FF0	F,FF0	FF,FF0	FFF,FF0	F,FFF,FF0	FF,FFF,FF0	FFF,FFF,FF0
2.	2 0's	-	-	F00	F,F00	FF,F00	FFF,F00	F,FFF,F00	FF,FFF,F00	FFF,FFF,F00
3.	3 0's	-	-	-	F,000	FF,000	FFF,000	F,FFF,000	FF,FFF,000	FFF,FFF,000
4.	4 0's	-	-	-	-	F0,000	FF0,000	F,FF0,000	FF,FF0,000	FFF,FF0,000
5.	5 0's	-	-	-	-	-	F00,000	F,F00,000	FF,F00,000	FFF,F00,000

Sr. No.	Number Range < → Last Hexadecimal Number with Successive↓	16^1	16^2	16^3	16^4	16^5	16^6	16^7	16^8	16^9
6.	6 0’s	-	-	-	-	-	-	F,000,000	FF,000,000	FFF,000,000
7.	7 0’s	-	-	-	-	-	-	-	F0,000,000	FF0,000,000
8.	8 0’s	-	-	-	-	-	-	-	-	F00,000,000

So their base *b* formulation is identical to the one in [11].

Formula 2 : If *n*, *r* and *b* are natural numbers with $b > 1$, then the last occurrence of *r* number successive of 0’s in base *b* numbers in range $1 \le m < b^n$ is

$$l = \begin{cases} - & \text{, if } r \ge n \\ b^n - b^r & \text{, if } r < n \end{cases}.$$

Remark : All the findings here have generalized the earlier ones in [6].

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