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EXPANSION FORMULAE INVOLVING A-FUNCTION

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ABSTRACT

In this paper, we establish some new some new expansion formulae involving A-function of two variables.

1. INTRODUCTION:

The subject of expansion formulae of generalized hypergeometric functions occupies a vital position in the literature of special functions. Certain two-dimensional expansion formulae of generalized hypergeometric functions participate major role in the growth of the theories of special functions and two-dimensional boundary value problems.

The A-function of one variable is defined by Gautam [2] and we will represent here in the following manner:

$$A^{m,\,n}_{p,\,q}\left[x\Big|^{((a_p,\,\alpha_p))}_{((b_q,\,\beta_q))}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\, ds \qquad (1)$$

where $i = \sqrt{(-1)}$ and

(i)

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(a_j + s\alpha_j) \prod_{j=1}^{n} \Gamma(1 - b_j - s\beta_j)}{\prod_{j=m+1}^{p} \Gamma(1 - a_j - s\alpha_j) \prod_{j=n+1}^{q} \Gamma(b_j + s\beta_j)} \qquad (2)$$

(ii) m, n, p and q are non-negative numbers in which $m \le p$, $n \le q$.

(iii) $x \neq 0$ and parameters a_j, α_j, b_k and β_k ($j = 1$ to p and $k = 1$ to q) are all complex.

The integral in the right hand side of is convergent if

(i) $x \neq 0, k = 0, h > 0, |\arg(ux)| < \pi h/2$

(ii) $x > 0, k = 0 = h, (\nu - \sigma\omega) < -1$

where

$$k = \operatorname{Im}\left(\sum_1^p \alpha_j - \sum_1^q \beta_j\right)$$

$$h = \operatorname{Re}\left(\sum_{j=1}^{m} \alpha_j - \sum_{j=m+1}^{p} \alpha_j + \sum_{j=1}^{n} \beta_j - \sum_{j=m+1}^{q} \beta_j\right) \quad (3)$$

$$u = \prod_1^p \alpha_j^{\alpha_j} \prod_1^q \beta_j^{\beta_j} \quad (4)$$

$$\nu = \operatorname{Re}\left(\sum_1^p a_j - \sum_1^q b_j\right) - (p - q)/2\,,$$

$$w = \operatorname{Re}\left(\sum_1^q \beta_j - \sum_1^p \alpha_j\right)$$

and $s = \sigma + it$ is on path L when $|t| \to \infty$.

In our investigation we shall need the following results:

From Shrivastava [4, p.174-176], we have

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(1+\sigma+n)\Gamma(-n-\sigma)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(1+n+\rho)}$$

$$.\sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha+n+k)\Gamma(1+\beta+n+k)\Gamma(\alpha+\beta+n+k-\rho-\sigma)}{k!\,\Gamma(1+\alpha+\beta+n+k-\sigma)\Gamma(\alpha+k-\rho-\sigma)}, \quad (5)$$

provided that $\operatorname{Re}(\rho + 1) > 0$, $\operatorname{Re}(1 + \sigma) > 0$, $\operatorname{Re}(-n - \sigma) > 0$, $\operatorname{Re}(1 + \alpha) > 0$, $\operatorname{Re}(\alpha + \beta + n + k - \rho - \sigma) > 0$.

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{(-1)^n 2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(1+\rho+n)\Gamma(-n-\rho)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(1+n+\sigma)} \times$$

$$.\sum_{k=0}^{\infty} \frac{\Gamma(1+\beta+n+k)\Gamma(1+\sigma+n+k)\Gamma(\alpha+\beta+n+k-\rho-\sigma)}{k!\,\Gamma(1+\alpha+\beta+n+k-\rho)\Gamma(\beta+k-\rho-\sigma)}, \quad (6)$$

provided that Re(σ + 1) > 0, Re(1 + ρ) > 0, Re (– n – ρ) > 0, Re (– ρ) > 0, Re (1 + β) > 0, Re(α + β + n + k – ρ – σ) > 0.

From Rainvile [3]:

$$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2 dx$$

$$=\frac{2^{\alpha+\beta+1}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}. \qquad (7)$$

2. MAIN INTEGRAL:

In this section, we shall establish following integrals:

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx$$

$$=\frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}\Gamma(1+\beta+n+k)$$

$$A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma,\delta),(1-\alpha-\beta-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+\rho+\sigma+n,\mu+\delta),(1+n+\rho,\mu)}^{(1+\rho,\mu),(1+\sigma+n,\delta),(a_j,\alpha_j)_{1,p},(-\alpha-\beta-n-k+\sigma,\delta),(1-\alpha-k+\rho+\sigma,\mu+\delta)}\right], \qquad (8)$$

provided that Re(1 + α) > 0, Re(1 + ρ + μ) > 0, Re(1 + σ + n + δ) > 0, Re(α + β + n + k – ρ – σ – (μ + δ)) > 0, Re(1 + n + k + ρ + μ) > 0, Re(– n – σ – δ) > 0, |arg (uz)| < ½ πh, where h and u are given in (3) and (4) respectively.

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx$$

$$=(-1)^n\frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)}{k!}\times$$

$$A_{p+5,q+4}^{m+3,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\rho,\mu),(1-\alpha-\beta-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+\rho+\sigma+n,\mu+\delta),(1+n+\sigma,\delta)}^{(1+\sigma,\delta),(1+\rho+n,\mu),(1+n+k+\sigma,\delta),(a_j,\alpha_j)_{1,p},(-\alpha-\beta-n-k+\rho,\mu),(1-\beta-k+\rho+\sigma,\mu+\delta)}\right], \qquad (9)$$

provided that Re(1 + β) > 0, Re(1 + σ + δ) > 0, Re(1 + ρ + n + μ) > 0, Re(α + β + n + k – ρ – σ – (μ + δ)) > 0, Re(1 + n + k + σ + μ) > 0, Re(– n – ρ – μ) > 0, |arg (uz)| < ½ πh, where h and u are given in (3) and (4) respectively.

Proof of (8):

To establish (8), replace the A-function by its equivalent counter integral as given in (1), we get

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)$$

$$.[\frac{1}{2\pi i}\int_L \theta(s)\,z^s(1-x)^{\mu s}(1+x)^{\delta s}ds]dx.$$

Change the order of integration which is valid under the given condition, we arrive at

$$\frac{1}{2\pi i}\int_L \theta(s)\,z^s\left[\int_{-1}^{1}(1-x)^{\rho+\mu s}\,(1+x)^{\sigma+\delta s}P_n^{(\alpha,\beta)}(x)dx\right]ds.$$

Now evaluate the inner integral with the help of (5) and finally interpret it with (1), we get (8).

The results (9) can be established easily in the view of (6) exactly on the same lines as given above.

3. **EXPANSION FORMULAE INVOLVING A-FUNCTION OF ONE VARIABLE:**

In this section, we established following expansion formulae involving A-function of one variable:

$$(1-x)^{\rho}(1+x)^{\sigma}A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$\times A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma+\beta,\delta),(1-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},}^{(1+\rho+\alpha,\mu),(1+\sigma+\beta+n,\delta),(a_j,\alpha_j)_{1,p},},\right.$$

$$\left.{}_{(2+\rho+\sigma+n+\alpha+\beta,\mu+\delta),(1+n+\rho+\alpha,\mu)}^{(-\alpha-n-k+\sigma,\delta),(1-k+\rho+\sigma+\beta,\mu+\delta)}\right], \qquad (10)$$

provided that Re(1 + α) > 0, Re(1 + β) > 0, Re(1 + ρ + α + μ) > 0, Re(1 + σ + β + n + δ) > 0, Re(n + k − ρ − σ − (μ + δ)) > 0, Re(− n − σ − δ) > 0, |arg (uz)| < ½ πh, where h and u are given in (3) and (4) respectively.

$$(1-x)^{\rho}(1+x)^{\sigma}A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(-1)^n(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\beta+n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$.\,A_{p+5,q+4}^{m+3,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\rho+\alpha,\mu),(1-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},}^{(1+\sigma+\beta,\delta),(1+\rho+n+\alpha,\mu),(1+n+k+\sigma+\beta,\delta),(a_j,\alpha_j)_{1,p},},\right.$$

$$\left.{}_{(2+\rho+\sigma+n+\alpha+\beta,\mu+\delta),(1+n+\sigma+\beta,\delta)}^{(-\beta-n-k+\rho,\mu),(1+\alpha-k+\rho+\sigma,\mu+\delta)}\right], \qquad (11)$$

provided that $Re(1+\alpha)>0$, $Re(1+\beta)>0$, $Re(1+\sigma+\beta+\delta)>0$, $Re(1+\rho+\alpha+n+\mu)>0$, $Re(n+k-\rho-\sigma-(\mu+\delta))>0$, $Re(1+n+k+\sigma+\beta+\mu)>0$, $Re(-n-\rho-\alpha-\mu)>0$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (3) and (4) respectively.

Proof:

To prove (10), consider

$$(1-x)^{\rho}(1+x)^{\sigma}A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{R=0}^{\infty} C_R\, P_R^{(\alpha,\beta)}(x). \qquad (12)$$

Equation (12) is valid, because in the left hand side the expression is continuous and is of bounded variation in the interval (− 1, 1). On multiplying both side of (12) by $(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$ and integrating between − 1 to 1 with respect to x, using relation (8) in left hand side, interchanging the order of integration and summation, which is valid under the condition [1, p.176(65)], using orthogonality property of Jacobi Polynomials (7), we get

$$C_n = \frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}.\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}$$

$$A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma+\beta,\delta),(1-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+\rho+\sigma+n+\alpha+\beta,\mu+\delta),(1+n+\rho+\alpha,\mu)}^{(1+\rho+\alpha,\mu),(1+\sigma+\beta+n,\delta),(a_j,\alpha_j)_{1,p},(-\alpha-n-k+\sigma,\delta),(1-k+\rho+\sigma+\beta,\mu+\delta)}\right], \qquad (13)$$

Further using (13) in (12), we get the relation (10).

Similarly, the results (11) can be established on lines by using the results (9) in place of (8).

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