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A STUDY ON INVERSE STRONG TERNARY GAMMA SEMIRING

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ABSTRACT

In this paper we introduce the notions of Strong Ternary Gamma Semiring (STΓ-Semiring). We prove that may result. We establish some relationship between the idempotent for both the addition and ternary multiplication. We prove in the case STΓ-Semiring, that the set of ternary multiplicative idempotent; $E^{[\ ]}(T)$ is closed under the ternary multiplication and so $(T,\Gamma,+,[\ \ ])$ is an orthodox ST Γ-Semiring.

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Key Words: Ternary Γ-Semiring, STΓ-Semiring, Idempotent, Orthodox STΓ-Semiring, Medial law, left inverse law.

Introduction:

The ring of integers has a great role in the theory of rings. The ternary operations are used to study the static hazards in combinational switching circuits by means of a suitable ternary switching algebra. The ternary operations appear also in the study of Quark model to explain the non-observability of isolated quarks as a phenomenon of algebraic confinement.

1. STΓ-Semiring:

Definition 1.1: Let T and Γ be two additive commutative semi groups. T is said to be a ***Strong Ternary Γ-Semiring*** or simply called ***STΓ-Semiring*** if there exist a mapping from T×Γ×T×Γ×T to T which maps $(x_1,\ \alpha, x_2,\ \beta, x_3) \rightarrow [x_1\alpha x_2\beta x_3]$ satisfying the conditions:

i) $[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]]$

ii)$[(a + b)\alpha c\beta d] = [a\alpha c\beta d] + [b\alpha c\beta d]$

iii) $[a(b + c)\beta d] = [a\alpha b\beta d] + [a\alpha c\beta d]$

iv) $[a\alpha b\beta(c + d)] = [a\alpha b\beta c] + [a\alpha b\beta d]$

v) $\left[a\alpha b\beta c\right]=\left[c\beta b\alpha a\right]$ for all $a, b, c, d \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Definition 1.2: A non-empty subset W of a strong ternary Γ-Semiring T is a strong ternary sub Γ-Semiring if and only if W + W ⊆ W and WΓWΓW ⊆ W.

Definition 1.3: A non-empty subset I of a STΓ-Semiring T is a ***left (resp. Lateral, right) ST Γ-Ideal*** of T if and only if I is additive sub semi group of T and $T\Gamma T\Gamma I \subseteq I$ (resp. $T\Gamma I\Gamma T \subseteq I,\ I\Gamma T\Gamma T \subseteq I$).

Definition 1.4: A non-empty subset I of a STΓ-Semiring T is a ***STΓ-Ideal*** of T Provided I is left, lateral and right STΓ-Ideal of T.

Definition 1.5: An element x of a STΓ-Semiring T is said to be ***ternary multiplicatively regular*** if there exist $y \in T$ and $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta x = x$ and $y\alpha x\beta y = y$. Then the element y is called a ***ternary multiplicative inverse*** of x.

Definition 1.6: An element x of a STΓ-Semiring T is said to be ***additively regula***r provided there exist $y \in T$ such that $x + y + x = x$ and $y + x + y = y$. Then the element y is called ***additive inverse*** of x.

Definition 1.7: A STΓ-Semiring T is said to be ***additive (resp. Ternary multiplicative) inverse STΓ-Semiring*** provided every element of T has a unique additive (resp. Ternary multiplicative) inverse.

Definition 1.8: Let T be an additive inverse STΓ-Semiring and a' denotes the unique inverse of a. We say that T satisfies the conditions (P), (Q) or (R) for all $a, b \in T$ and $\alpha, \beta \in \Gamma$.

1. (P) $a\alpha(a+a')\beta a = a+a'$, 2. (Q) $a\alpha a\beta(b+b') = (b+b')\alpha a\beta a = a\alpha(b+b')\beta a$

3. (R) $a + a\alpha(b+b')\beta a = a$.

Definition 1.9: Let T be a STΓ-Semiring. We denote by $E^{+}(T) = \{a \in T / a+a = a\}$ the set of additive idempotent and by $E^{[\ ]}(T) = \{e \in T, \alpha, \beta \in \Gamma / e\alpha e\beta e = e\}$ the set of all ternary multiplicative idempotent.

Note that $E^{+}(T)$ is a ternary multiplicative STΓ-Ideal of T.

2. Inverses in STΓ-Semiring:

Theorem 2.1: Any STΓ-Semiring T satisfies the medial law: for all $p_i, q_i, r_i \in T$

$$(p_1\Gamma p_2\Gamma p_3)\Gamma(q_1\Gamma q_2\Gamma q_3)\Gamma(r_1\Gamma r_2\Gamma r_3) = (p_1\Gamma q_1\Gamma p_3)\Gamma(p_2\Gamma q_2\Gamma r_2)\Gamma(r_1\Gamma q_3\Gamma r_3)$$

Proof: for all $p_i, q_i, r_i \in T$ we have

$$\begin{aligned}
&(p_1\Gamma p_2\Gamma p_3)\Gamma(q_1\Gamma q_2\Gamma q_3)\Gamma(r_1\Gamma r_2\Gamma r_3) = p_1\Gamma\left[p_2\Gamma p_3\Gamma(q_1\Gamma q_2\Gamma q_3)\right]\Gamma(r_1\Gamma r_2\Gamma r_3)\\
&= p_1\left[(p_2\Gamma p_3\Gamma q_1)\Gamma q_2\Gamma q_3)\right]\Gamma(r_1\Gamma r_2\Gamma r_3) = p_1\left[(q_1\Gamma p_3\Gamma p_2)\Gamma q_2\Gamma q_3)\right]\Gamma(r_1\Gamma r_2\Gamma r_3)\\
&= (p_1\Gamma q_1\Gamma p_3)\Gamma(p_2\Gamma q_2\Gamma q_3)\Gamma(r_1\Gamma r_2\Gamma r_3) = (p_1\Gamma q_1\Gamma p_3)\Gamma[p_2\Gamma q_2\Gamma(q_3\Gamma r_1\Gamma r_2)\Gamma r_3]\\
&= (p_1\Gamma q_1\Gamma p_3)\Gamma[p_2\Gamma q_2\Gamma(r_2\Gamma r_1\Gamma q_3)\Gamma r_3] = (p_1\Gamma q_1\Gamma p_3)\Gamma(p_2\Gamma q_2\Gamma r_2)\Gamma(r_1\Gamma q_3\Gamma r_3)
\end{aligned}$$

Theorem 2.2: Let T be an additive inverse STΓ-Semiring

1) **If e ∈ $E^{[\,]}$(T) and** e' **∈ $E^{[\,]}$(T), then e =** e'
2) **e ∈ $E^{[\,]}$(T), then** e' **∈ $E^{[\,]}$(T) and e +** e' **∈ $E^{[\,]}$(T)**
3) $E^{[\,]}(T)\Gamma E^{[\,]}(T)\Gamma E^{[\,]}(T) \subseteq E^{[\,]}(T)$ **and in this case T is called a STΓ-Orthodox Semiring.**

Proof: 1) $e' = e'\Gamma e'\Gamma e' = e\Gamma e\Gamma e = e$ so $e = e'$.

2) $e'\Gamma e'\Gamma e' = e'\Gamma e'\Gamma(e'+e+e')' = e'\Gamma e'\Gamma e' + e'\Gamma e'\Gamma e + e'\Gamma e'\Gamma e'$ and

$e'\Gamma e'\Gamma e + e'\Gamma e'\Gamma e' + e'\Gamma e'\Gamma e = e'\Gamma e'\Gamma(e+e'+e) = e'\Gamma e'\Gamma e$. Therefore $e'\Gamma e'\Gamma e$ is an additive inverse of $e'\Gamma e'\Gamma e'$. Here T is additive inverse STΓ-Semiring $(e'\Gamma e'\Gamma e')' = e'\Gamma e'\Gamma e$. In other hand $e'\Gamma e'\Gamma e = e'\Gamma(e'+e+e')\Gamma e = e'\Gamma e'\Gamma e + e'\Gamma e\Gamma e + e'\Gamma e'\Gamma e$ and then $(e'\Gamma e'\Gamma e)' = e\Gamma e\Gamma e'$. Using the unicity, $(e'\Gamma e'\Gamma e')' = e'\Gamma e'\Gamma e = (e\Gamma e\Gamma e')'$. Finally using same expression, we have $(e\Gamma e\Gamma e)' = (e'\Gamma e'\Gamma e')$ but as $e \in E^{[\,]}(T)$ $e' = (e\Gamma e\Gamma e)' = e'\Gamma e'\Gamma e'$ and hence $e' \in E^{[\,]}(T)$ and

$(e+e') + (e+e') = (e+e'+e) + e' = e+e'$

3) $i, j, k \in E^{[\,]}$(T) from theorem 2.1,

$$\begin{aligned}
&(i\Gamma j\Gamma k)\Gamma(i\Gamma j\Gamma k)\Gamma(i\Gamma j\Gamma k) = (i\Gamma i\Gamma k)\Gamma(j\Gamma j\Gamma j)\Gamma(i\Gamma k\Gamma k) = (i\Gamma i\Gamma k)\Gamma j\Gamma(i\Gamma k\Gamma k) = i\Gamma i\Gamma(k\Gamma j\Gamma i)\Gamma k\Gamma k\\
&= i\Gamma i\Gamma(i\Gamma j\Gamma k)\Gamma k\Gamma k = (i\Gamma i\Gamma i)\Gamma j\Gamma(k\Gamma k\Gamma k) = i\Gamma j\Gamma k
\end{aligned}$$

Therefore $i\Gamma j\Gamma k \in E^{[\,]}(T)$. Hence $E^{[\,]}(T)\Gamma E^{[\,]}(T)\Gamma E^{[\,]}(T) \subseteq E^{[\,]}(T)$.

Theorem 2.3: Let T be a STΓ-Semiring. If $a', b', c' \in T$ **denotes additive inverse of *a, b, c* then** $a'\Gamma b'\Gamma c$ **is an additive inverse of** $a'\Gamma b'\Gamma c'$.

Proof: $a'\Gamma b'\Gamma c + a'\Gamma b'\Gamma c' + a'\Gamma b'\Gamma c = a'\Gamma b'\Gamma(c\Gamma c'\Gamma c) = a'\Gamma b'\Gamma c$ and

$$a'\Gamma b'\Gamma c' + a'\Gamma b'\Gamma c + a'\Gamma b'\Gamma c' = a'\Gamma b'\Gamma(c'\Gamma c\Gamma c') = a'\Gamma b'\Gamma c'.$$

Therefore $a'\Gamma b'\Gamma c$ is an additive inverse of $a'\Gamma b'\Gamma c'$.

Corollary 2.4: Let T be an additive inverse STΓ-Semiring. Then for any element *a, b, c* in T, the following conditions are hold:

1) $a'\Gamma b'\Gamma c = a\Gamma b'\Gamma c' = a'\Gamma b\Gamma c'$
2) $(a\Gamma b\Gamma c)' = a'\Gamma b\Gamma c = a\Gamma b'\Gamma c = a\Gamma b\Gamma c'$

Proof: 1) As proved in the theorem 2.3, we can show that $a\Gamma b'\Gamma c'$ and $a'\Gamma b'\Gamma c$ are also two additive inverses of $a'\Gamma b'\Gamma c'$ and the conclusion follows from the uniqueness of additive inverse of any element of T.

2) Proof is trivial.

Corollary 2.5: Let T be an additive inverse STΓ-Semiring. Then for any element *a* in T, the following conditions are hold:

1) $(a')^* = (a^*)'$
2) $(a + a' + a)^* = a^* + (a')^* + a^*$

Proof: 1) $(a^*)'\ \Gamma a'\ \Gamma(a^*)' = [[[a^*\Gamma a\Gamma a^*]']']' = [[[a^*]']']' = [a^*]'$ by using uniqueness. In the other hand $a'\ \Gamma[a^*]'\ \Gamma a' = [[[a\Gamma a^*\Gamma a]']']' = [[[a]']']' = a'$. Therefore, the result follows.

2) $(a + a' + a)^* = a^* = a^* + (a')^* + a^*$ from the previous equality.

Theorem 2.6: 1) If T be a STΓ-Semiring and T satisfies the conditions (P) and (R), then for any additive inverse x' ∈ T, we have $x + x + x' = x$.

2) If in addition $x \in E^{[\ ]}(T)$ and it has a ternary multiplicative inverse then 3*x* = *x*.

Proof: 1) From (R), we have $x + x\Gamma(x + x')\Gamma x = x$. But from (P) as $x\Gamma(x + x')\Gamma x = x + x'$, we deduce that $x + x\Gamma(x + x')\Gamma x = x + (x + x') = x + x + x'$ and hence $x + x + x' = x$.

2) From (R) we also have

$x = x + x\Gamma(x + x^*)\Gamma x \Leftrightarrow x + x\Gamma x\Gamma x + x\Gamma x^*\Gamma x = x + x\Gamma x\Gamma x + x = x\Gamma x\Gamma x = x \Rightarrow 3x = x$.

Since $x\Gamma x\Gamma x = x$ and hence 3*x* = *x*.

Definition 2.7: Let T be a STΓ-Semiring and A be a subset of T. We define

$$A^l = \{t \in T / t\Gamma r\Gamma s \subseteq E^+(T), \forall r, s \in A\}$$

$$A^m = \{t \in T / r\Gamma t\Gamma s \subseteq E^+(T), \forall r, s \in A\}$$

$A^r = \{t \in T / r\Gamma s\Gamma t \subseteq E^+(T), \forall r,s \in A\}$

Theorem 2.8: Let T be a STΓ-Semiring and I be a subset of T, then $E^+(T) \subset I^l \cap I^m \cap I^r$.

Proof: Let $a \in E^+(T)$ and $b, c \in$ I, then $(a\Gamma b\Gamma c)+(a\Gamma b\Gamma c)=(a+a)\Gamma b\Gamma c = a\Gamma b\Gamma c$

and $(b\Gamma c\Gamma a)+(b\Gamma c\Gamma a)=b\Gamma c\Gamma(a+a)=b\Gamma c\Gamma a$.

Similarly, $(c\Gamma a\Gamma b)+(c\Gamma a\Gamma b)=c\Gamma(a+a)\Gamma b = c\Gamma a\Gamma b$.

Therefore, $a\Gamma b\Gamma c \subseteq E^+(T)$, $b\Gamma c\Gamma a \subseteq E^+(T)$ and $c\Gamma a\Gamma b \subseteq E^+(T)$, finally $a \in I^l \cap I^m \cap I^r$.

Definition 2.9: A STΓ-Semiring T is said to be a ***cyclic STΓ-Semiring*** if in the definition 1.1, the condition (V) is replaced by $(V)'$ $a\Gamma b\Gamma c = b\Gamma c\Gamma a = c\Gamma a\Gamma b$.

Theorem 2.10: Let T be a cycle STΓ-Semiring. If A is a left STΓ-Ideal, then 1)$T\Gamma T\Gamma A^l \subseteq A^r, T\Gamma T\Gamma A^l \subseteq A^m$ 2) $A^r\Gamma T\Gamma T \subseteq A^l, A^r\Gamma T\Gamma T \subseteq A^m$.

Proof: Let $a \in A^l$, $t,t' \in T$ and $x,y \in A$, then

$x\Gamma y\Gamma(t\Gamma t'\Gamma a) = a\Gamma x\Gamma(y\Gamma t\Gamma t') = a\Gamma x\Gamma(t\Gamma t'\Gamma y)$, since A is a left STΓ-Ideal and hence $t\Gamma t'\Gamma y \subseteq A$ and as $a \in A^l, x \in A$ then $a\Gamma x\Gamma(y\Gamma t\Gamma t') \subseteq E^+(T)$ and so $t\Gamma t'\Gamma a \subseteq A^r$.

Again $x\Gamma(t\Gamma t'\Gamma a)\Gamma y = (x\Gamma t\Gamma t')\Gamma a\Gamma y = a\Gamma y\Gamma(x\Gamma t\Gamma t') = a\Gamma y\Gamma(t\Gamma t'\Gamma x)$, since A is a left STΓ-Ideal and hence $t\Gamma t'\Gamma x \subseteq A$ and as $a \in A^l, y \in A$, then $a\Gamma y\Gamma(t\Gamma t'\Gamma x) \subseteq E^+(T)$ and so $t\Gamma t'\Gamma x \subseteq A^m$ and hence $T\Gamma T\Gamma A^l \subseteq A^m$.

Now let $a \in A^r$, $t,t' \in T$ and $x,y \in A$, then

$$(a\Gamma t\Gamma t')\Gamma x\Gamma y = x\Gamma y\Gamma(a\Gamma t\Gamma t') = (x\Gamma y\Gamma a)\Gamma t\Gamma t'$$
$$= [x\Gamma y\Gamma a + x\Gamma y\Gamma a]\Gamma t\Gamma t' = (a\Gamma t\Gamma t')\Gamma x\Gamma y + (a\Gamma t\Gamma t')\Gamma x\Gamma y$$

Since $a \in A^r$ and then $x\Gamma y\Gamma a \subseteq E^+(T)$. Finally, $(a\Gamma t\Gamma t') \subseteq A^l$.

Again, $x\Gamma(a\Gamma t\Gamma t')\Gamma y = (a\Gamma t\Gamma t')\Gamma y\Gamma x = y\Gamma x\Gamma(a\Gamma t\Gamma t')$
$$= (y\Gamma x\Gamma a)\Gamma t\Gamma t' = (y\Gamma x\Gamma a + y\Gamma x\Gamma a)\Gamma t\Gamma t'$$
$$= (a\Gamma t\Gamma t')\Gamma y\Gamma x + (a\Gamma t\Gamma t')\Gamma y\Gamma x$$

Since $a \in A^r$ and then $y\Gamma x\Gamma a \subseteq E^+(T)$. Finally, $(a\Gamma t\Gamma t') \subseteq A^m$. Therefore $A^r\Gamma T\Gamma T \subseteq A^m$.

Theorem 2.11: Let T be a cyclic STΓ-Semiring. If + is commutative and A is a ternary Γ-ideal, then $A^l \cap A^m \cap A^r$ is a ternary Γ-ideal.

Proof: Let $b \in A^l \cap A^m \cap A^r, t, t' \in T$ and $x, y \in I$.

$$(b\Gamma t\Gamma t')\Gamma x\Gamma y + (b\Gamma t\Gamma t')\Gamma x\Gamma y = x\Gamma y\Gamma(b\Gamma t\Gamma t') + x\Gamma y\Gamma(b\Gamma t\Gamma t')$$
$$= (x\Gamma y\Gamma b)\Gamma t\Gamma t' + (x\Gamma y\Gamma b)\Gamma t\Gamma t' = [(x\Gamma y\Gamma b)+(x\Gamma y\Gamma b)]\Gamma t\Gamma t'$$
$$= (x\Gamma y\Gamma b)\Gamma t\Gamma t' = x\Gamma y\Gamma(b\Gamma t\Gamma t') = (b\Gamma t\Gamma t')\Gamma x\Gamma y$$

Since $x\Gamma y\Gamma b \subseteq E^{+}(T)$. Then $b\Gamma t\Gamma t' \subseteq A^{l}$ $\rightarrow (1)$

$$x\Gamma y(b\Gamma t\Gamma t') = (b\Gamma t\Gamma t')\Gamma x\Gamma y = (t\Gamma t'\Gamma x)\Gamma y\Gamma b = t\Gamma t'\Gamma(x\Gamma y\Gamma b)$$
$$= t\Gamma t'(x\Gamma y\Gamma b + x\Gamma y\Gamma b)$$

Since $x\Gamma y\Gamma b \subseteq E^{+}(T)$. So $x\Gamma y\Gamma(b\Gamma t\Gamma t') \subseteq E^{+}(T)$ and hence $b\Gamma t\Gamma t' \subseteq A^{r} \rightarrow (2)$.

Now $x\Gamma(b\Gamma t\Gamma t')\Gamma y = y\Gamma x\Gamma(b\Gamma t\Gamma t') = (y\Gamma x\Gamma b)\Gamma t\Gamma t'$. Since $y\Gamma x\Gamma b \subseteq E^{+}(T)$. Therefore, $x\Gamma(b\Gamma t\Gamma t')\Gamma y \subseteq E^{+}(T)$ and hence $b\Gamma t\Gamma t' \subseteq A^{m}$ $\rightarrow (3)$

From (1), (2) and (3) we have $(A^{l} \cap A^{m} \cap A^{r})\Gamma T\Gamma T \subseteq A^{l} \cap A^{m} \cap A^{r}$.

In the other hand, as for all $b \in A^{l} \cap A^{m} \cap A^{r}$, $t, t' \in T$ one has $t\Gamma t'\Gamma b = b\Gamma t\Gamma t'$ and using the previous facts we get $T\Gamma T\Gamma(A^{l} \cap A^{m} \cap A^{r}) = (A^{l} \cap A^{m} \cap A^{r})\Gamma T\Gamma T \subseteq A^{l} \cap A^{m} \cap A^{r}$.

Now let $x, y \in A^{l} \cap A^{m} \cap A^{r}$ and $a, b \in A$. Then

$$(x+y)\Gamma a\Gamma b + (x+y)\Gamma a\Gamma b = x\Gamma a\Gamma b + y\Gamma a\Gamma b + x\Gamma a\Gamma b + y\Gamma a\Gamma b$$
$$=(x\Gamma a\Gamma b + x\Gamma a\Gamma b) + (y\Gamma a\Gamma b + y\Gamma a\Gamma b)$$
$$= x\Gamma a\Gamma b + y\Gamma a\Gamma b$$

Since $x, y \in A^{l}$ and hence $x + y \in A^{l}$, with the same arguments we can easily prove that $x + y \in A^{m}$ and $x + y \in A^{r}$. Finally, $A^{l} \cap A^{m} \cap A^{r} + A^{l} \cap A^{m} \cap A^{r} \subseteq A^{l} \cap A^{m} \cap A^{r}$.

Remark 2.12: It is clear that if in addition, T has a ternary multiplicative identity say 1, then $T\Gamma T\Gamma(A^{l} \cap A^{m} \cap A^{r}) = (A^{l} \cap A^{m} \cap A^{r})\Gamma T\Gamma T = A^{l} \cap A^{m} \cap A^{r}$.

Theorem 2.13: Let T be a ternary multiplicative inverse STΓ-Semiring. If $x \in E^{[\ ]}(T)$, then $x^{*} = x$ and hence $x^{*} \in E^{[\ ]}(T)$.

***Proof*:** For every $x \in E^{[\ ]}(T)$, we have

$$(x\Gamma x\Gamma x)\Gamma(x^{*}\Gamma x\Gamma x)\Gamma(x\Gamma x\Gamma x) = x\Gamma x\Gamma[(x\Gamma x^{*}\Gamma x)\Gamma x\Gamma(x\Gamma x\Gamma x)]$$
$$= x\Gamma x\Gamma[x\Gamma x\Gamma(x\Gamma x\Gamma x)] = x\Gamma x\Gamma[x\Gamma x\Gamma x] = x\Gamma x\Gamma x.$$

In the other hand;

$$(x^{*}\Gamma x\Gamma x)\Gamma(x\Gamma x\Gamma x)\Gamma(x^{*}\Gamma x\Gamma x) = x^{*}\Gamma x\Gamma[(x\Gamma x\Gamma x)\Gamma x\Gamma(x\Gamma x\Gamma x)] = x^{*}\Gamma x\Gamma(x\Gamma x\Gamma x) = x^{*}\Gamma x\Gamma x$$

So the ternary multiplicative inverse of $x\Gamma x\Gamma x$ is $x^{*}\Gamma x\Gamma x$ but as $x\Gamma x\Gamma x = x$ and x^{*} is the unique ternary multiplicative inverse of a then $x^{*} = x^{*}\Gamma x\Gamma x$.

With the same considerations we can easily prove that $x\Gamma x^{*}\Gamma x$ and $x\Gamma x\Gamma x^{*}$ are ternary multiplicative inverses of $x\Gamma x\Gamma x$ and by the uniqueness of any ternary multiplicative inverse, we get $(x\Gamma x\Gamma x)^{*} = x\Gamma x^{*}\Gamma x = x\Gamma x\Gamma x^{*} = x^{*}\Gamma x\Gamma x.$ But $x\Gamma x^{*}\Gamma x = x$ and then $x^{*} = x\Gamma x^{*}\Gamma x = x$.

Theorem 2.14: Let T be a STΓ-Semiring. If each element has a multiplicative inverse then for all $a,b,c \in T$; $c^*\Gamma b^*\Gamma a^*$ is a ternary multiplicative inverse of $a\Gamma b\Gamma c$, where a^*, b^*, c^* are respectively some ternary multiplicative inverses of *a, b, c*.

Proof: If *a, b, c* ∈T and a^*, b^*, c^* are as required, then

$$\begin{aligned}
(a\Gamma b\Gamma c)\Gamma(c^*\Gamma b^*\Gamma a^*)\Gamma(a\Gamma b\Gamma c) &= a\Gamma b\Gamma[c\Gamma(c^*\Gamma b^*\Gamma a^*)\Gamma(a\Gamma b\Gamma c)] \\
&= a\Gamma b\Gamma[c\Gamma c^*\Gamma(b^*\Gamma a^*\Gamma a)\Gamma b\Gamma c)] = a\Gamma b\Gamma[c\Gamma c^*\Gamma[c\Gamma b\Gamma(b^*\Gamma a^*\Gamma a)] \\
&= a\Gamma b\Gamma(c\Gamma c^*\Gamma c)\Gamma b\Gamma(b^*\Gamma a^*\Gamma a)] = a\Gamma b\Gamma[c\Gamma b\Gamma(b^*\Gamma a^*\Gamma a)] \\
&= a\Gamma b\Gamma[(c\Gamma b\Gamma b^*)\Gamma a^*\Gamma a] = a\Gamma b\Gamma[a\Gamma a^*\Gamma(c\Gamma b\Gamma b^*)] = (a\Gamma b\Gamma a)\Gamma a^*\Gamma(c\Gamma b\Gamma b^*) \\
&= (c\Gamma b\Gamma b^*)\Gamma a^*\Gamma(a\Gamma b\Gamma a) = (c\Gamma b\Gamma b^*)\Gamma(a^*\Gamma a\Gamma b)\Gamma a) \\
&= (c\Gamma b\Gamma b^*)\Gamma(b\Gamma a\Gamma a^*)\Gamma a = c\Gamma(b\Gamma b^*\Gamma b)\Gamma(a\Gamma a^*\Gamma a) = c\Gamma b\Gamma a = a\Gamma b\Gamma c.
\end{aligned}$$

In the other hand by replacing a,b,c,a^*,b^*,c^* respectively by c^*,b^*,a^*,c,b,a in the previous relation we have $(c^*\Gamma b^*\Gamma a^*)\Gamma(a\Gamma b\Gamma c)\Gamma(c^*\Gamma b^*\Gamma a^*) = c^*\Gamma b^*\Gamma a^*$.

So $c^*\Gamma b^*\Gamma a^*$ is an inverse of $a\Gamma b\Gamma c$.

Definition 2.15: Let *a* be an element of a STΓ-Semiring T, we define the powers of *x* as:
$(x\Gamma)^2 x = x\Gamma x\Gamma x$, $(x\Gamma)^4 x = x\Gamma x\Gamma x\Gamma x$, $(x\Gamma)^{2n} x = (x\Gamma)^{2n-1} x\Gamma x$ for all $n > 1$.

Theorem 2.16: Let T be a STΓ-Semiring. If x^* is an inverse of x then $x^*\Gamma x^*\Gamma(x\Gamma)^2 x = x\Gamma x\Gamma x^*$ and for all $n \geq 2$; $x^*\Gamma x^*\Gamma(x\Gamma)^{2n} x = (x\Gamma)^{2n} x$.

Proof: for all *x* ∈T, we have

$$\begin{aligned}
x^*\Gamma x^*\Gamma(x\Gamma)^2 x &= (x^*\Gamma x^*\Gamma x)\Gamma x\Gamma x = (x\Gamma x^*\Gamma x^*)\Gamma x\Gamma x = x\Gamma x^*\Gamma(x^*\Gamma x\Gamma x) \\
&= x\Gamma x^*\Gamma(x\Gamma x\Gamma x^*) = (x\Gamma x^*\Gamma x)\Gamma x\Gamma x^* = x\Gamma x\Gamma x^*
\end{aligned}$$

In other hand $x^*\Gamma x^*\Gamma(x\Gamma)^4 x = [x^*\Gamma x^*\Gamma(x\Gamma)^2 x\Gamma x\Gamma x] = x\Gamma x\Gamma x^*\Gamma x\Gamma x$

$$= x\Gamma(x\Gamma x^*\Gamma x)\Gamma x = x\Gamma x\Gamma x = (x\Gamma)^2 x$$

And so by induction for any $n \geq 2$,

$$x^*\Gamma x^*\Gamma(x\Gamma)^{2n} x = x^*\Gamma x^*\Gamma(x\Gamma)^{2n-1} x = (x\Gamma)^{2n-1} x\Gamma x = (x\Gamma)^{2n} x$$

Theorem 2.17: Let T be a STΓ-Semiring. If x^* is an inverse of *x* and *n* is such that $n \geq 1$ for odd natural number *n*; then $x \in E^+(T) \Rightarrow (x\Gamma)^2 x \subseteq E^+(T) \Leftrightarrow (x\Gamma)^{2n} \subseteq E^+(T)$.

Proof: The first implication is trivial. For the equivalence; since $(x\Gamma)^2 x \subseteq E^+(T)$ and $(x\Gamma)^{2n+2} x + (x\Gamma)^{2n+2} x = [(x\Gamma)^{2n+1} + (x\Gamma)^{2n+1}]x\Gamma x$. The direct implication can then be done by induction on $n \geq 1$.

The converse can be done by a decreasing induction: In one hand we have $(x\Gamma)^{2n} \subseteq E^{+}(T)$.

In the other hand, suppose that $(x\Gamma)^{2n-p-1}x \subseteq E^{+}(T)$ $\forall 1 \le p \le 2n-5$ then;

$$(x\Gamma)^{2n-p-3}x\Gamma x + (x\Gamma)^{2n-p-3}x\Gamma x = x^*\Gamma x^*\Gamma(x\Gamma)^{2n-p-1}x + x^*\Gamma x^*\Gamma(x\Gamma)^{2n-p-1}x$$
$$= x^*\Gamma x^*\Gamma[(x\Gamma)^{2n-p-1}x] = (x\Gamma)^{2n-p-3}x.$$

So by taking $P = 2n - 5$, we get $(x\Gamma)^2 x \subseteq E^{+}(T)$.

Theorem 2.18: If T is a cycle STΓ-Semiring then;

1) for all *n* ≥1, *x*, *y* and *z* ∈ T;

$$(x\Gamma)^{2n}x\Gamma(y\Gamma)^{2n}y\Gamma(z\Gamma)^{2n}z = (x\Gamma y\Gamma z)\Gamma(x\Gamma y\Gamma z)\Gamma[(x\Gamma)^{2n-2}\Gamma(z\Gamma)^{2n-2}\Gamma(y\Gamma)^{2n-2}y] \qquad \text{(S)}$$

2) If x^* is a ternary multiplicative inverse of x then $(x^*\Gamma)^{2n}x^*$ is a ternary multiplicative inverse of $(x\Gamma)^{2n}x$.

***Proof*:** Since T is a cycle ST**Γ**-Semiring; $(a\Gamma)^{2n}a = (a\Gamma)^{2n-1}a\Gamma a = a\Gamma a\Gamma(a\Gamma)^{2n-2}a$ then

$$
\begin{aligned}
(x\Gamma)^{2n}x\Gamma(y\Gamma)^{2n}y\Gamma(z\Gamma)^{2n}z &= [x\Gamma x\Gamma(x\Gamma)^{2n-2}x]\Gamma[y\Gamma y\Gamma(y\Gamma)^{2n-2}y]\Gamma[z\Gamma z\Gamma(z\Gamma)^{2n-2}z] \\
&= x\Gamma x\Gamma[((x\Gamma)^{2n-2}x)\Gamma[y\Gamma y\Gamma(y\Gamma)^{2n-2}y]\Gamma[z\Gamma z\Gamma(z\Gamma)^{2n-2}z]] \\
&= x\Gamma x\Gamma[((x\Gamma)^{2n-2}x)\Gamma y\Gamma y)\Gamma((y\Gamma)^{2n-2}y)\Gamma(z\Gamma z\Gamma(z\Gamma)^{2n-2}z)] \\
&= x\Gamma x\Gamma[(y\Gamma(x\Gamma)^{2n-2}x\Gamma y)\Gamma(y\Gamma)^{2n-2}y\Gamma(z\Gamma z\Gamma(z\Gamma)^{2n-2}z)] \\
&= (x\Gamma x\Gamma y)\Gamma[(x\Gamma)^{2n-2}x\Gamma y\Gamma(y\Gamma)^{2n-2}y]\Gamma(z\Gamma z\Gamma(z\Gamma)^{2n-2}z) \\
&= (x\Gamma y\Gamma x)\Gamma[y\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x]\Gamma(z\Gamma z\Gamma(z\Gamma)^{2n-2}z) \\
&= (x\Gamma y\Gamma x)\Gamma y\Gamma[(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x\Gamma z]\Gamma z\Gamma(z\Gamma)^{2n-2}z) \\
&= (x\Gamma y\Gamma x)\Gamma y\Gamma[z\Gamma(z\Gamma)^{2n-2}z\Gamma[(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x\Gamma z]] \\
&= [(x\Gamma y\Gamma x)\Gamma y\Gamma z]\Gamma(z\Gamma)^{2n-2}z\Gamma[(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x\Gamma z] \\
&= [(x\Gamma y\Gamma x)\Gamma y\Gamma z]\Gamma(z\Gamma)^{2n-2}z\Gamma[z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= [x\Gamma y\Gamma(z\Gamma x\Gamma y)]\Gamma(z\Gamma)^{2n-2}z\Gamma[z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= [(x\Gamma y\Gamma z)\Gamma x\Gamma y]\Gamma(z\Gamma)^{2n-2}z\Gamma[z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= (x\Gamma y\Gamma z)\Gamma[x\Gamma y\Gamma(z\Gamma)^{2n-2}z]\Gamma[z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= [(x\Gamma y\Gamma z)\Gamma x\Gamma y]\Gamma(z\Gamma)^{2n-2}z\Gamma[z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= [((x\Gamma y\Gamma z)\Gamma x\Gamma y)\Gamma(z\Gamma)^{2n-2}z\Gamma z]\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x \\
&= [z\Gamma((x\Gamma y\Gamma z)\Gamma x\Gamma y)\Gamma(z\Gamma)^{2n-2}z]\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x \\
&= z\Gamma((x\Gamma y\Gamma z)\Gamma x\Gamma y)\Gamma[(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= z\Gamma(x\Gamma y\Gamma z)\Gamma[x\Gamma y\Gamma[(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x]] \\
&= (z\Gamma x\Gamma y)\Gamma z\Gamma[x\Gamma y\Gamma[(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x]] \\
&= (x\Gamma y\Gamma z)\Gamma z\Gamma[x\Gamma y\Gamma[(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x]] \\
&= (x\Gamma y\Gamma z)\Gamma(z\Gamma x\Gamma y)\Gamma[(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= (x\Gamma y\Gamma z)\Gamma(x\Gamma y\Gamma z)\Gamma[(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y\Gamma(x\Gamma)^{2n-2}x] \\
&= (x\Gamma y\Gamma z)\Gamma(x\Gamma y\Gamma z)\Gamma[(x\Gamma)^{2n-2}x\Gamma(z\Gamma)^{2n-2}z\Gamma(y\Gamma)^{2n-2}y]
\end{aligned}
$$

2) By induction on $n \geq 1$ and by replacing in the equality (S), y, z respectively by x^*, x. We get for $n = 1$;

$$
\begin{aligned}
(x\Gamma)^2 x\Gamma(x^*\Gamma)^2 x^*\Gamma(x\Gamma)^2 x &= (x\Gamma x^*\Gamma x)\Gamma(x\Gamma x^*\Gamma x)\Gamma(x\Gamma x\Gamma x^*) \\
&= x\Gamma x\Gamma(x\Gamma x^*\Gamma x) = x\Gamma x\Gamma x = (x\Gamma)^2 x
\end{aligned}
$$

Suppose that $(x\Gamma)^{2n-2}x\Gamma(x^*\Gamma)^{2n-2}x^*\Gamma(x\Gamma)^{2n-2}x = (x\Gamma)^{2n-2}x$, then

$$
\begin{aligned}
&((x\Gamma)^{2n}x)\Gamma((x^*\Gamma)^{2n}x^*\Gamma((x\Gamma)^{2n}x) = (x\Gamma x^*\Gamma x)\Gamma(x\Gamma x^*\Gamma x)\Gamma((x\Gamma)^{2n-2}x)\Gamma((x^*\Gamma)^{2n-2}x^*\Gamma((x\Gamma)^{2n-2}x) \\
&= x\Gamma x\Gamma((x\Gamma)^{2n-2}x)\Gamma((x^*\Gamma)^{2n-2}x^*\Gamma((x\Gamma)^{2n-2}x) = x\Gamma x\Gamma((x\Gamma)^{2n-2}x) = (x\Gamma)^{2n-2}x\Gamma x\Gamma x = (x\Gamma)^{2n}x.
\end{aligned}
$$

Theorem 2.19: Any ternary multiplicative inverse of an element of a STΓ-Semiring is unique.

Proof: Let x, y be two ternary multiplicative inverses of an element $a \in$T. Then

$x = x\Gamma a\Gamma x = x\Gamma(a\Gamma y\Gamma a)\Gamma x = x\Gamma a\Gamma(y\Gamma a\Gamma x) = x\Gamma a\Gamma(x\Gamma a\Gamma y) = (x\Gamma a\Gamma x)\Gamma a\Gamma y = x\Gamma a\Gamma y$ and by inverting x in y and vice-versa we get $y = y\Gamma a\Gamma x$ and since $x\Gamma a\Gamma y = y\Gamma a\Gamma x$. We get the uniqueness of the multiplicative inverse.

Conclusion: In this paper mainly we studied about Strong ternary Γ-semirings.

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