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ON p^α-CLOSED SETS, p^α-CONTINUITY AND α-p-ALMOST COMPACTNESS FOR CRISP SUBSETS OF A FUZZY TOPOLOGICAL SPACE

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ABSTRACT

This paper is a continuation of [1]. A new class of crisp subsets has been introduced and studied here by which α-p-almost compactness of a space X (endowed with a fuzzy topology) has been inherited. Again, p^α-continuous function between two fuzzy topological spaces has been introduced under which α-p-almost compactness for crisp subsets remains invariant.

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1. Introduction

It is clear from literature that a good many researchers have engaged themselves for introducing different types of compactness in fuzzy setting by using the concept of fuzzy cover [2] of a fuzzy topological space (fts, for short) [2]. Afterwards, in 1978 Gantner et al. [3] generalized the idea of fuzzy cover by introducing the concept of α-shading which has paved a new direction for generalizing different types of compactness. In [3], α-compactness for crisp subsets has been introduced. Using the idea of α-shading, in [1]

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α-p-almost compactness for crisp subsets (i.e., ordinary subsets) in a fuzzy topological space has been introduced and studied.

2. Preliminaries

A fuzzy set A in an fts X means a function from a non-empty set X to the closed interval $I = [0, 1]$ of the real line, i.e., $A \in I^X$ [6]. A crisp subset A of an fts X means an ordinary subset A of X, i.e., $A \subseteq X$, where the underlying structure on X is a fuzzy topology τ. The support of a fuzzy set A in X will be denoted by $suppA$ [6] and is defined by $suppA = \{x \in X: A(x) \neq 0\}$. A fuzzy point in X with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. For two fuzzy sets A and B in X, we write $A \leq B$ if $A(x) \leq B(x)$, for all $x \in X$, while we write AqB if A is quasi-coincident (q-coincident, for short) with B [5], i.e., $A(x) + B(x) > 1$, for some $x \in X$. The negation of these two statements are written as $A \nleq B$ and $A\bar{q}B$ respectively. The complement of a fuzzy set A in an fts X is denoted by $1_X \backslash A$ and is defined by $(1_X \backslash A)(x) = 1 - A(x)$, for all $x \in X$ [6]. For a fuzzy set A in an fts X, clA and $intA$ stand for fuzzy closure and fuzzy interior of A in X respectively [2].

3. Fuzzy Preopen and Fuzzy Preclosed Sets and α- p-Almost Compact Space

Let us recall some definitions for ready references.

Definition 3.1 [4]. A fuzzy set A in an fts X is said to be fuzzy preopen if $A \leq int(clA)$. The complement of a fuzzy preopen set is called fuzzy preclosed.

Definition 3.2 [4]. For a fuzzy set A in an fts X, fuzzy preclosure of A to be denoted by $pclA$ and is defined to be the smallest fuzzy preclosed set containing A, i.e., $pclA = \wedge \{B: A \leq B$ and B is fuzzy preclosed$\}$. A fuzzy set A in X is fuzzy preclosed iff $A = pclA$.

Definition 3.3 [4]. For a fuzzy set A in an fts X, the fuzzy preinterior of A to be denoted by $pintA$ and is defined to be the union of all those fuzzy preopen sets contained in A, i.e., $pintA = \vee \{B: B \leq A$ and B is fuzzy preopen in $X\}$. A fuzzy set A in X is fuzzy prepen iff $A = pintA$.

Definition 3.4 [3]. Let A be a crisp subset of an fts X. A collection $\mathcal{U}$ of fuzzy sets in X is called an α-shading (where $0< \alpha< 1$) of A if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$. Taking $A = X$, we arrive at the definition of α-shading of an fts X, as proposed by Gantner et al. [3].

If the members of an α-shading $\mathcal{U}$ of A (or of X) are fuzzy preopen sets in X, then $\mathcal{U}$ is called a fuzzy preopen α-shading of A (resp., of X).

Definition 3.5 [1]. Let X be an fts and A, a crisp subset of X. A is called α-p-almost compact if each fuzzy preopen α-shading of A has a finite p-proximate α-subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{pclU : U \in \mathcal{U}_0\}$ is again an α-shading of A.

If $A = X$, in addition, then X is called an α-p-almost compact space.

4. p^{α}-Closed Sets : Some Properties

In this section a new class of crisp subsets in an fts X is defined as follows :

Definition 4.1. Let (X, τ) be an fts and $A \subseteq X$. A point $x \in X$ is said to be p^{α}-limit point of A if for every fuzzy preopen set U in X with $U(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(pclU)(y) > \alpha$. The set of all p^{α}-limit points of A will be denoted by $[A]_p^{\alpha}$.

The p^{α}-closure of A, to be denoted by $p^{\alpha} - clA$, is defined by $p^{\alpha} - clA = A \cup [A]_p^{\alpha}$.

Definition 4.2. A crisp subset A of an fts X is said to be p^{α}-closed if it contains all its p^{α}-limit points. Any subset A of X is called p^{α}-open if $X \setminus A$ is p^{α}-closed.

Remark 4.3. Definition 4.1 shows that for any $A \subseteq X$, $A \subseteq p^{\alpha} - clA$ and $p^{\alpha} - clA = A$ if and only if $[A]_p^{\alpha} \subseteq A$. Also A is p^{α}-closed if and only if $p^{\alpha} - clA = A$. It is also clear that $A \subseteq B \subseteq X \implies [A]_p^{\alpha} \subseteq [B]_p^{\alpha}$.

Theorem 4.4. In an α-p-almost compact space X, a p^{α}-closed subset A is α-p-almost compact set.

Proof. Let A $(\subseteq X)$ be p^{α}-closed in an α-p-almost compact space X. Then for any $x \notin A$, x is not a p^{α}-limit point of A and so there is a fuzzy preopen set U_x in X such that $U_x(x) > \alpha$ and $(pclU_x)(y) \leq \alpha$, for every $y \in A$. Consider the collection $\mathcal{U} = \{U_x : x \notin A\}$. Let us consider a fuzzy preopen α-shading $\mathcal{V}$ of A. Clearly $\mathcal{U} \cup \mathcal{V}$ is fuzzy preopen α-shading of X. Since X is α-p-almost compact space, there is a finite subcollection $\{V_1, V_2, \ldots, V_n\}$ of $\mathcal{U} \cup \mathcal{V}$ such that for every $t \in X$, there exists V_i $(1 \leq i \leq n)$ such that $(pclV_i)(t) > \alpha$.

For every member U_x of $\mathcal{U}$, $(pclU_x)(y) \leq \alpha$, for every $y \in A$. So if this collection contains any member of $\mathcal{U}$, we omit it and hence we get the result.

To achieve the converse of Theorem 4.4, we define the following.

Definition 4.5. An fts (X, τ) is said to be α-p-Urysohn if for any two distinct points x, y of X, there exist a fuzzy open set U and a fuzzy preopen set V in X with $U(x) > \alpha$, $V(y) > \alpha$ and $\min\big((pclU)(z), (pclV)(z)\big) \leq \alpha$, for each $z \in X$.

Theorem 4.6. An α-p-almost compact set in an α- p-Urysohn space X is p^{α}-closed.

Proof. Let A be α-p-almost compact set and $x \in X \setminus A$. Then for each $y \in A, x \neq y$. As X is α- p-Urysohn, there exist a fuzzy open set U_y and a fuzzy preopen set V_y in X such that $U_y(x) > \alpha$, $V_y(y) > \alpha$ and $\min\left(\left(pclU_y\right)(z), \left(pclV_y\right)(z)\right) \leq \alpha$, for all $z \in X$ ….. (1).

Then $\mathcal{U} = \{V_y : y \in A\}$ is a fuzzy preopen α-shading of A and so by α-p-almost compactness of A, there is finitely many points $y_1, y_2, \dots, y_n$ of A such that $\mathcal{U}_0 = \{pclV_{y_1}, pclV_{y_2}, \dots, pclV_{y_n}\}$ is again an α-shading of A. Now $U = U_{y_1} \cap U_{y_2} \cap \dots. \cap U_{y_n}$ being a fuzzy open set is a fuzzy preopen set in X such that $U(x) > \alpha$. In order to show that A to be p^{α}-closed, it now suffices to show that $(pclU)(y) \leq \alpha$, for each $y \in A$. If possible, let for some $z \in A$, $(pclU)(z) > \alpha$. Then as $z \in A$, we have $\left(pclV_{y_k}\right)(z) > \alpha$, for some k $(1 \leq k \leq n)$. Again $\left(pclU_{y_k}\right)(z) > \alpha$. Hence $\min\left(\left(pclU_{y_k}\right)(z), \left(pclV_{y_k}\right)(z)\right) < \alpha$, contradicting (1).

Corollary 4.7. In an α-p-almost compact, α-p-Urysohn space X, a subset A of X is α-p-almost compact if and only if it is p^{α}-closed.

Theorem 4.8. In an α-p-almost compact space X, every cover of X by p^{α}-open sets has a finite subcover.

Proof. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a cover of X by p^{α}-open sets of X. Then for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $X \setminus U_x$ is p^{α}-closed, there exists a fuzzy preopen set V_x in X such that $V_x(x) > \alpha$, but $(pclV_x)(y) \leq \alpha$, for each $y \in X \setminus U_x$ ….. (1).

Then $\{V_x : x \in X\}$ forms a fuzzy preopen α-shading of X. Since X is α-p-almost compact space, there exists a finite subset $\{x_1, x_2, \dots., x_n\}$ of X such that $\{pclV_{x_i} : i = 1, 2, \dots, n\}$ is again an α-shading of X … (2).

We claim that $\{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y \in X \setminus \bigcup_{i=1}^{n} U_{x_i} = \bigcap_{i=1}^{n}(X \setminus U_{x_i})$. Then by (1), $(pclV_{x_i})(y) \le \alpha$ for $i = 1, 2, \dots, n$ and so $(\bigcup_{i=1}^{n} pclV_{x_i})(y) \le \alpha$, contradicting (2).

Theorem 4.9. Let (X, τ) be an fts. If X is α-p-almost compact, then every collection of p^α-closed sets in X with finite intersection property has non-empty intersection.

Proof. Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a collection of p^α-closed sets in an α-p-almost compact space X having finite intersection property. If possible, let $\bigcap_{i \in \Lambda} F_i = \varphi$. Then $X \setminus \bigcap_{i \in \Lambda} F_i = \bigcup_{i \in \Lambda}(X \setminus F_i) = X \Longrightarrow \mathcal{U} = \{X \setminus F_i : i \in \Lambda\}$ is an p^α-open cover of X. Then by Theorem 4.8, there is a finite subset Λ_0 of Λ such that $\bigcup_{i \in \Lambda_0}(X \setminus F_i) = X \Longrightarrow \bigcap_{i \in \Lambda_0} F_i = \varphi$, a contradiction.

5. p^α-Continuity : Some Applications

In this section, a new type of function has been introduced under which α-p-almost compactness remains invariant.

Definition 5.1. Let X, Y be fts's. A function $f : X \to Y$ is said to be p^α-continuous if for each point $x \in X$ and each fuzzy preopen set V in Y with $V(f(x)) > \alpha$, there exists a fuzzy preopen set U in X with $U(x) > \alpha$ such that $pclU \le f^{-1}(pclV)$.

Theorem 5.2. If $f : X \to Y$ is p^α-continuous (where X, Y are, as usual, fts's), then the following are true :

(a) $f([A]_p^\alpha) \subseteq [f(A)]_p^\alpha$, for every $A \subseteq X$,

(b) $[f^{-1}(A)]_p^\alpha \subseteq f^{-1}([A]_p^\alpha)$, for every $A \subseteq Y$,

(c) for each p^α-closed set A in Y, $f^{-1}(A)$ is p^α-closed in X,

(d) for each p^α-open set A in Y, $f^{-1}(A)$ is p^α-open in X.

Proof (a). Let $x \in [A]_p^\alpha$ and U be any fuzzy preopen set in Y with $U(f(x)) > \alpha$. Then there is a fuzzy preopen set V in X with $V(x) > \alpha$ and $pclV \le f^{-1}(pclU)$. Now $x \in [A]_p^\alpha$ and V be a fuzzy preopen set in X with $V(x) > \alpha \Longrightarrow pclV(x_0) > \alpha$ for some $x_0 \in A \setminus \{x\} \Longrightarrow \alpha < pclV(x_0) \le (f^{-1}(pclU))(x_0) = (pclU)(f(x_0))$ where $f(x_0) \in f(A) \setminus \{f(x)\} \Longrightarrow f(x) \in [f(A)]_p^\alpha$. Thus (a) follows.

(b) By (a), $f([f^{-1}(A)]_p^\alpha) \subseteq [ff^{-1}(A)]_p^\alpha \subseteq [A]_p^\alpha \Longrightarrow [f^{-1}(A)]_p^\alpha \subseteq f^{-1}([A]_p^\alpha)$.

(c) We have $[A]_p^\alpha = A$. By (b), $[f^{-1}(A)]_p^\alpha \subseteq f^{-1}([A]_p^\alpha) = f^{-1}(A) \Rightarrow [f^{-1}(A)]_p^\alpha = f^{-1}(A) \Rightarrow f^{-1}(A)$ is p^α-closed set in X.

(d) Follows from (c).

Theorem 5.3. Let X, Y be fts's and $f : X \to Y$ be fuzzy p^α-continuous. If A $(\subseteq X)$ is α-p-almost compact, then so is $f(A)$ in Y.

Proof. Let $\mathcal{V} = \{V_i : i \in \Lambda\}$ be a fuzzy preopen α-shading of $f(A)$, where A is α-p-almost compact set in X. For each $x \in A$, $f(x) \in f(A)$ and so there exists $V_x \in \mathcal{V}$ such that $V_x(f(x)) > \alpha$. As f is fuzzy p^α-continuous, there exists a fuzzy preopen set U_x in X such that $U_x(x) > \alpha$ and $f(pclU_x) \leq pclV_x$. Then $\{U_x : x \in A\}$ is a fuzzy preopen α-shading of A. By α-p-almost compactness of A, there are finitely many points $a_1, a_2, \dots, a_n$ in A such that $\{pclU_{a_i} : i = 1, 2, \dots, n\}$ is again an α-shading of A. We claim that $\{pclV_{a_i} : i = 1, 2, \dots, n\}$ is again an α-shading of $f(A)$. Infact, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. As $x \in A$, there is some U_{a_j} (for some $j, 1 \leq j \leq n$) such that $\left(pclU_{a_j}\right)(x) > \alpha$ and hence $\left(pclV_{a_j}\right)(y) \geq f\left(pclU_{a_j}\right)(y) \geq pclU_{a_j}(x) > \alpha$.

We now define a function under which p^α-closedness of a set remains invariant.

Definition 5.4. Let X, Y be fts's. A function $f : X \to Y$ is said to be fuzzy preopen if $f(A)$ is fuzzy preopen in Y whenever A is fuzzy preopen in X.

Remark 5.5. For a fuzzy preopen function $f : X \to Y$, every fuzzy preclosed set A in X, $f(A)$ is fuzzy preclosed in Y.

Theorem 5.6. If $f : (X, \tau) \to (Y, \tau_1)$ is a bijective fuzzy preopen function, then the image of a p^α-closed set in (X, τ) is p^α-closed in (Y, τ_1).

Proof. Let A be a p^α-closed set in (X, τ) and let $y \in Y \setminus f(A)$. Then there exists a unique $z \in X$ such that $f(z) = y$. As $y \notin f(A)$, $z \notin A$. Now, A is p^α-closed in X, z is not a p^α-limit point of A and so there exists a fuzzy preopen set V in X such that $V(z) > \alpha$, but $pclV(t) \leq \alpha$, for each $t \in A$ … (1).

As f is fuzzy preopen, $f(V)$ is a fuzzy preopen set in Y and also $(f(V))(y) = V(z) > \alpha$. Let $s \in f(A)$. Then there is a unique $s_0 \in A$ such that $f(s_0) = s$. As f is bijective and fuzzy preopen, by Remark 5.5, $pclf(V) \leq f(pclV)$. Then $(pclf(V))(s) \leq f(pclV)(s) = pclV(s_0) \leq \alpha$, by (1). Thus y is not a p^α-limit point of $f(A)$. Hence the proof.

From Theorem 5.2(c) and Theorem 5.6, it follows that

Corollary 5.7. Let $f : X \to Y$ be a fuzzy p^{α}-continuous, bijective and fuzzy preopen function. Then A is p^{α}-closed in Y if and only if $f^{-1}(A)$ is p^{α}-closed in X.

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