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STEIN-RULE RESTRICTED RIDGE REGRESSION ESTIMATOR

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ABSTRACT

Stein-rule and ridge estimators have been extensively used for estimating the coefficient vector in a regression model. These estimators lead to an improvement in the risk properties of the ordinary least squares (OLS) estimator. Instead of using one or the other estimator, both of them may be appropriately combined. We introduce an alternative estimator that combines the approaches followed in obtaining the restricted Stein-rule estimation and the ridge regression estimation. A Monte Carlo simulation is performed to compare the behavior of the proposed estimator.

Keywords: Stein-rule estimator; ridge regression estimator; restricted estimator; quadratic loss function.

1. Introduction

Let us consider the linear regression model

$$y = X\beta + u , \tag{1}$$

where y is an $n\times 1$ vector of observations on the dependent variable, X is an $n\times p$ nonstochastic full column rank matrix of observations on independent variables, β is a $p\times 1$ vector of unknown regression coefficients and u is an $n\times 1$ vectors disturbances assumed to follow a normal distribution $N(0,\sigma^2 V^{-1})$. The elements of the matrix $V = V(\theta)$ are functions of a $q\times 1$ parameter vector θ, which belongs to an open subset of the q dimensional Eucledian space.

We assume the exact linear restrictions binding the regression coefficients vector β of m linearly independent rows as follows:

$$r = R\beta \tag{2}$$

where r is an $m \times 1$ vector and R is an $m \times p$ matrix of rank m ($< p$).

The feasible generalized restricted least squares (FGRLS) estimator is given by

$$\breve{\beta}_R = \breve{\beta} + (X'\hat{V}X)^{-1} R'[R(X'\hat{V}X)^{-1} R']^{-1}(r - R\breve{\beta}), \tag{3}$$

where $\hat{V} = V(\hat{\theta})$, $\hat{\theta}$ is a consistent estimator of θ, and

$$\breve{\beta} = (X'\hat{V}X)^{-1} X'\hat{V}y \tag{4}$$

is the feasible generalized least squares (FGLS) estimator of β (see, Chaturvedi, et al. (2001)).

Chaturvedi et al. (2001) considered the estimators $\bar{\beta}_{RS}$ and $\breve{\beta}_{RS}$ which they are the "Stein-Like" restricted estimators as follows:

$$\bar{\beta}_{RS} = \breve{\beta}_R - \frac{a}{n} \frac{(y - X\breve{\beta})'\hat{V}(y - X\breve{\beta})}{\breve{\beta}'X'\hat{V}X\breve{\beta}} \hat{S}\,\breve{\beta}, \tag{5}$$

where $\hat{S} = \mathrm{I}_p - (X'\hat{V}X)^{-1} R'[R(X'\hat{V}X)^{-1} R']^{-1} R$. One may further consider replacing $\breve{\beta}$ by $\breve{\beta}_R$ in Equation (5), and the following estimator results:

$$\breve{\beta}_{RS} = \left[\mathrm{I}_p - \frac{a}{n} \frac{(y - X\breve{\beta}_R)'\hat{V}(y - X\breve{\beta}_R)}{\breve{\beta}_R' X'\hat{V}X\breve{\beta}_R} \hat{S} \right] \breve{\beta}_R, \tag{6}$$

where both estimators $\bar{\beta}_{RS}$ and $\breve{\beta}_{RS}$ satisfied the restrictions (2). $\bar{\beta}_{RS}$ and $\breve{\beta}_{RS}$ collapse to the estimators considered by Srivastava and Srivastava (1984) when $V^{-1} = \mathrm{I}_n$. The estimators of Srivastava and Chandra (1991) result when $V^{-1} = \mathrm{I}_n$ and the disturbances are not necessarily normally distributed.

Since both estimators $\bar{\beta}_{RS}$ and $\breve{\beta}_{RS}$ are not derived under quadratic loss subject to the binding constraints. So, it is misleading to call $\bar{\beta}_{RS}$ and $\breve{\beta}_{RS}$ Stein-rule estimators. In order to make Stein-rule estimation meaningful, Chaturvedi, et al. (2001) considered the following transformation, Rao (1973, p. 231) or Kariya (1979):

$$\tilde{y} = \tilde{X}\gamma + u \tag{7}$$

where $\tilde{X} = XN$, $N = \mathrm{I}_p - R'(RR')^{-1} R$, $\gamma = N\beta$, $\tilde{\gamma} = R'(RR')^{-1} r$, and

$$\begin{aligned} \tilde{y} &= y - X\tilde{\gamma} \\ &= y - X(\mathrm{I}_p - N)\beta. \end{aligned} \tag{8}$$

For combining a set of linear restrictions with the model, they premultiplied Equation (2) with $XR'(RR')^{-1}$ and then subtracted it from Equation (1), leading to

$$y - R'(RR')^{-1}\ r = X[\mathrm{I}_p - R'(RR')^{-1}R]\beta + u\ . \tag{9}$$

which is equivalent to model (7).

Note that the matrix $\tilde{X}'\hat{V}\tilde{X}$ is singular. Now, let A^{+} denote the generalized (Moore-Penrose) inverse of a matrix A. Then, in model (7), the FGLS estimator of γ is given by

$$\hat{\gamma} = (NX'\hat{V}XN)^{+}\, NX'\hat{V}\tilde{y}\,, \tag{10}$$

so that the restricted regression estimator of β is

$$\hat{\beta}_R = \hat{\gamma} + \tilde{\gamma}\ . \tag{11}$$

Now, using $\hat{\beta}_R$ and by applying Stein-rule to model (7), Chaturvedi, et al. (2001) obtained the following restricted Stein-rule estimator for β,

$$\hat{\beta}_{RS} = \left[1 - \frac{a}{n-p+\alpha}\,\frac{(\tilde{y}-\tilde{X}\hat{\gamma})'\hat{V}(\tilde{y}-\tilde{X}\hat{\gamma})}{\hat{\gamma}'N'X'\hat{V}XN\hat{\gamma}}\right]\hat{\gamma} + \tilde{\gamma}, \tag{12}$$

where α is any scalar parameter such that $n-p+\alpha$ is positive. Obviously, for $a=0$ the restricted Stein-rule estimator reduces to the estimator $\hat{\beta}_R$. Also, since $R(N'X'\hat{V}XN)=0$, using the properties of generalized inverse , it can be verified that $\hat{\beta}_{RS}$ satisfied the prior restriction (2).

Since the calculation of the generalized (Moore-Penrose) inverse of a matrix is very difficult, it will be used the ridge regression estimator, Hoerl and Kennard (1970), instead of the FGLS estimator $\hat{\gamma}$ in model (7) as follows

$$\hat{\gamma}_k = (N'X'\hat{V}XN + k\mathrm{I})^{-1}\, N'X'\hat{V}\tilde{y}\,, \quad k \geq 0 \tag{13}$$

also, by replacing $\hat{\gamma}_k$ with $\hat{\gamma}$ in Equation (11), we define new estimator as follows:

$$\hat{\beta}_R(k) = \hat{\gamma}_k + \tilde{\gamma}\ . \tag{14}$$

So, the new Stein type restricted ridge regression estimator is defined as

$$\hat{\beta}_{RS}(k) = \left[1 - \frac{a}{n-p+\alpha}\,\frac{(\tilde{y}-\tilde{X}\hat{\gamma}_k)'\hat{V}(\tilde{y}-\tilde{X}\hat{\gamma}_k)}{\hat{\gamma}_k'N'X'\hat{V}XN\hat{\gamma}_k}\right]\hat{\gamma}_k + \tilde{\gamma}, \tag{15}$$

$\hat{\beta}_{RS}(k)$ satisfied the prior restriction (2).

Since it is difficult to make theoretical comparisons among the estimators defined in Equations (12), (14) and (15) under a quadratic loss function, we will investigate the performance of these estimators using a Monte Carlo simulation study in Sec. 2.

2. A Monte Carlo Study

In this section, we will discuss the simulation study to compare the performances of the $\hat{\beta}_{RS}$, $\hat{\beta}_{R}(k)$ and $\hat{\beta}_{RS}(k)$ estimators. MATLAB is used for the simulation experiment. Following McDonald and Galarneau (1975) and Kibria (2003), the explanatory variables are generated by

$$x_{ij} = (1-\gamma^2)^{1/2} z_{ij} + \gamma z_{i\,p+1}\ ,\ i = 1,2,..., n, \quad j = 1,2,..., p \tag{16}$$

where z_{ij} are independent standard normal pseudo-random numbers, γ is specified so that the correlation between any two explanatory variables is given by γ^2. Following Kibria (2003), three different sets of correlation are considered, corresponding to $\gamma = 0.7, 0.8, 0.9$. The explanatory variables are then standardized so that $X'X$ is in the correlation form. Following Chaturvedi, et al. (2001), data are generated using the following orthonormal model with $p = 4$ and $n = 20$:

$$y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + u_i; \quad i = 1,\ldots,20 \tag{17}$$

where the u_i's are assumed to be generated by either an AR(1) process ($u_i = \rho u_{i-1} + \varepsilon_i$) or a MA(1) process ($u_i = \varepsilon_i - \rho\varepsilon_{i-1}$); where ρ is taken as -0.8, -0.4, 0.0, 0.4, 0.8, and $\varepsilon_i \sim IN(0,1)$. Also, $a = p-3$ and $\alpha = 2$ are taken like this.

Following Kaçıranlar et al. (2011), two different types of restrictions are used to see how the number of restrictions affects the mean square error (mse) values. The first type of restriction about the parameters is taken as $\beta_1 + \beta_2 + \cdots + \beta_p = \frac{p(p+1)}{2} \times \tau$. For this restriction, one may write $R = (1\ 1\ \ldots\ 1)$, and $r = \frac{p(p+1)}{2} \times \tau$. Since the true parameter vector is $\beta = (1\ 2\ \ldots\ p)'$, the sum given in the restriction is equal to $\frac{p(p+1)}{2} \times \tau$. The usage of τ allows one to control whether the restriction is true or not, and if it not true, to control the difference between the right and left hand sides of the restriction, τ is selected as 1, 1.05, or 1.10, and it is fixed constant throughout the replications. Since the left hand side of the restriction is equal to $\frac{p(p+1)}{2}$, it is obvious that the restriction $\beta_1 + \beta_2 + \cdots + \beta_p = \frac{p(p+1)}{2} \times \tau$ is true when $\tau = 1$. However, when τ is chosen as 1.05 (or 1.10), the right hand side of the restriction is equal to $\frac{p(p+1)}{2} \times 1.05$ (or

$\frac{p(p+1)}{2}\times 1.10$), while the left side remains $\frac{p(p+1)}{2}$, which leads to a 5% (or 10%) deviation on the right hand side of the restriction. Since R, r, β, and τ are all defined at the beginning of the simulations and fixed through the replications, the restrictions are still nonstochastic. For the second type of restriction R and r are chosen as

$$R_{\frac{p}{2}\times p} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \text{ and } r_{\frac{p}{2}\times 1} = \begin{pmatrix} 3 \\ 7 \\ \vdots \\ 2p-1 \end{pmatrix} \times \tau$$

where the number of restrictions is $m = p/2$. The function and values of τ are still the same as given in the first restriction.

The approaches of Hoerl et al. (1975) are used to specify the values of k. The estimates are denoted as $\hat{k}_{HK} = \frac{\hat{\sigma}^2_{FGLS}}{\sum_{i=1}^{p}\tilde{\gamma}_i^2}$, $\hat{k}_{HKB} = \frac{p\hat{\sigma}^2_{FGLS}}{\sum_{i=1}^{p}\tilde{\gamma}_i^2}$ and $\hat{k}_{LW} = \frac{p\hat{\sigma}^2_{FGLS}}{\sum_{i=1}^{p}\lambda_i\tilde{\gamma}_i^2}$ respectively, where $\hat{\sigma}^2_{FGLS}$ is the FGLS estimate of error variance and λ_i is the eigenvalues of the matrix $\tilde{X}'\tilde{X}$. The estimate of the scalar mse is used to compare the estimators, and it is defined as

$$EMSE(\hat{\beta}^*) = \frac{1}{5000}\sum_{mci=1}^{5000}(\hat{\beta}^* - \beta)'(\hat{\beta}^* - \beta),$$

where $\hat{\beta}^*$ is the $\hat{\beta}_{RS}$, $\hat{\beta}_R(k)$ or $\hat{\beta}_{RS}(k)$ estimate of β for the mci-th replication.

The results of the simulation study are summarized in Tables 1–4.

Under the first and the second restrictions when they are true or not, we have the following comments:

-The restrictions in this simulation study make the $\hat{\beta}_{RS}(k)$ and $\hat{\beta}_R(k)$ have smaller mse values relative to the $\hat{\beta}_{RS}$ in general.

-When the $\hat{\beta}_R(k)$ and $\hat{\beta}_{RS}(k)$ are compared for the same value of k, it is necessary that the $\hat{\beta}_{RS}(k)$ has smaller mse values than that of the $\hat{\beta}_R(k)$ for some k's and the $\hat{\beta}_R(k)$ has smaller mse values than that of the $\hat{\beta}_{RS}(k)$ for the other k's of the same situation.

-When the $\hat{\beta}_{RS}$ and $\hat{\beta}_{RS}(k)$ are compared, it is necessary that the $\hat{\beta}_{RS}(k)$ has smaller mse values than that of the $\hat{\beta}_{RS}$ in general.

-When the $\hat{\beta}_{RS}$ and $\hat{\beta}_R(k)$ are compared, it is necessary that the $\hat{\beta}_R(k)$ has smaller mse values than that of the $\hat{\beta}_{RS}$ in general.

-Under AR(1) and MA(1), we have that as γ increases, the scalar mses of the these estimators also increase as expected (e.g. for $\hat{\rho} = -0.8$, the scalar mses of the these estimators at $\gamma = 0.8$ are larger than the scalar mses of the these estimators at $\gamma = 0.7$).

-For γ *=0.7,0.8,0.9,* while $\hat{\rho}$ is increasing as a value, the estimated scalar mses of these estimators are decreasing (e.g. the scalar mses of the these estimators at $\hat{\rho} = -0.8$ are larger than the scalar mses of the these estimators at $\hat{\rho} = 0.8$). Also, when τ is increasing, the estimated scalar mses of these estimators are increasing (e.g. the scalar mses of the these estimators at $\tau = 1$-restrictions are true-are larger than the scalar mses of the these estimators at $\tau = 1.05$-restrictions are true-). Moreover, the scalar mses of these estimators for the first type restrictions are larger than the scalar mses of these estimators for the second type restrictions.

Note: For each row in Tables 1-4, the best estimator with the lowest mse value is bolded.

3. Conclusion

We introduce an alternative estimator that combines the Stein-rule and the ridge estimators which have been extensively used for estimating the coefficient vector in a regression model. And then we investigate the performance of the defined estimator using a Monte Carlo simulation study in which the results show in general that $\hat{\beta}_{RS}(k)$ and $\hat{\beta}_R(k)$ have smaller mse values relative to the $\hat{\beta}_{RS}$ under the first and the second restrictions when they are true or not and when the error term follows any of AR(1) or MA(1) model.

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Table 1. Restriction type 1, under $AR(1)$

τ	ρ	γ	$\hat{\beta}_{RS}$	$\hat{\beta}_{R}(k)$			$\hat{\beta}_{RS}(k)$		
			---------	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$
1	-0.8	0.7	1.6481	1.6946	1.5452	1.8121	1.4856	**1.3385**	1.6266
		0.8	2.2662	2.2169	1.8276	2.5224	1.9704	**1.6928**	2.2110
		0.9	4.0484	4.1241	3.1514	4.6477	3.2605	**2.5077**	3.7793
	-0.4	0.7	0.5424	0.5566	0.5978	0.5634	**0.5407**	0.6137	0.5413
		0.8	0.7555	0.7923	0.8724	0.7974	**0.7472**	0.8561	0.7521
		0.9	1.3648	1.3852	1.4152	1.4657	**1.3100**	1.5111	1.3380
	0.0	0.7	0.2232	0.2280	0.2635	0.2237	0.2290	0.2674	**0.2230**
		0.8	0.3161	0.3230	0.3824	0.3169	0.3275	0.3977	**0.3159**
		0.9	0.5980	0.6060	0.7214	0.5988	0.6353	0.8279	**0.5980**
	0.4	0.7	0.1248	0.1247	0.1784	0.1253	0.1258	0.1440	**0.1246**
		0.8	0.1761	0.1866	0.2547	0.1793	0.1780	0.2117	**0.1760**
		0.9	0.3299	0.3265	0.3791	0.3330	0.3364	0.4343	**0.3297**
	0.8	0.7	0.0171	0.0267	0.1352	**0.0170**	0.0175	0.0207	0.0172
		0.8	0.0243	0.0274	0.1445	**0.0242**	0.0249	0.0311	0.0243
		0.9	0.0461	0.0463	0.0632	**0.0457**	0.0483	0.0683	0.0464
1.05	-0.8	0.7	1.7198	1.7638	1.6093	1.8846	1.5573	**1.4188**	1.6982
		0.8	2.3341	2.2787	1.8960	2.5935	2.0326	**1.7577**	2.2779
		0.9	4.1039	4.1600	3.1413	4.7115	3.2938	**2.5324**	3.8276
	-0.4	0.7	0.6164	0.6349	0.6974	0.6354	0.6231	0.7198	**0.6162**
		0.8	0.8293	0.8710	0.9720	0.8708	0.8299	0.9674	**0.8270**
		0.9	1.4334	1.4705	1.5186	1.5444	1.3852	1.6274	**1.4075**
	0.0	0.7	0.2927	0.2951	0.3285	**0.2915**	0.3034	0.3583	0.2936
		0.8	0.3889	0.4005	0.4854	**0.3880**	0.4091	0.5084	0.3912
		0.9	0.6811	0.6991	0.8626	**0.6801**	0.7414	1.0039	0.6951
	0.4	0.7	0.1913	0.1939	0.2583	0.1917	0.1944	0.2209	**0.1912**
		0.8	0.2501	0.2623	0.3480	0.2528	0.2554	0.3045	**0.2500**
		0.9	0.4343	0.4343	0.5157	0.4374	0.4495	0.5894	**0.4306**
	0.8	0.7	0.0854	0.0932	0.2032	**0.0852**	0.0856	0.0898	0.0854
		0.8	0.1093	0.1193	0.2140	0.1091	**0.1087**	0.1152	0.1091
		0.9	0.2021	**0.1955**	0.2015	0.2018	0.1965	0.2129	0.1999
1.10	-0.8	0.7	1.9372	1.9767	1.7979	2.1074	1.7728	**1.6511**	1.9150
		0.8	2.5432	2.4887	2.1214	2.8167	2.2322	**1.9800**	2.4846
		0.9	4.2767	4.3191	3.2412	4.9219	3.4293	**2.7246**	3.9841
	-0.4	0.7	0.8349	0.8604	0.9524	0.8538	0.8526	0.9882	**0.8307**
		0.8	1.0489	1.0971	1.2274	1.0931	1.0639	1.2631	**1.0482**
		0.9	1.6449	1.7021	1.7775	1.7755	**1.6216**	2.0095	1.6223
	0.0	0.7	0.4998	0.5009	0.5347	**0.4965**	0.5194	0.6084	0.5014
		0.8	0.6053	0.6256	0.7619	**0.6020**	0.6436	0.8099	0.6099
		0.9	0.9269	0.9627	1.2238	**0.9224**	1.0442	1.4931	0.9552
	0.4	0.7	0.3904	0.3967	0.4858	0.3906	0.3974	**0.3903**	0.3909
		0.8	0.4707	0.4810	0.5531	0.4730	0.4833	**0.4706**	0.4717
		0.9	0.7417	0.7461	0.8900	0.7481	0.7754	**0.7320**	0.7463

	0.8	0.7	0.2904	0.2975	0.4253	**0.2901**	0.2904	0.3010	0.2903
		0.8	0.3638	0.3718	0.4614	**0.3522**	0.3590	0.3728	0.3628
		0.9	0.6606	0.6302	**0.6051**	0.6660	0.6216	0.6397	0.6468

Table 2. Restriction Rype 1, under $MA(1)$

τ	ρ	γ	$\hat{\beta}_{RS}$	$\hat{\beta}_R(k)$			$\hat{\beta}_{RS}(k)$		
			---------	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$
1	-0.8	0.7	2.4551	2.5264	2.2261	2.6796	2.3251	**2.1516**	2.4396
		0.8	3.4085	3.5161	3.0020	3.7735	3.1812	**2.8646**	3.3701
		0.9	6.2235	6.5638	5.3788	7.0373	5.6417	**4.7855**	6.0406
	-0.4	0.7	0.5252	0.5271	0.6694	0.5391	0.5345	0.6451	**0.5251**
		0.8	0.7336	0.7471	0.8031	0.7643	0.7488	0.9309	**0.7335**
		0.9	1.3327	1.3630	1.4074	1.4214	1.3617	1.7744	**1.3299**
	0.0	0.7	0.2232	0.2280	0.2635	0.2237	0.2290	0.2674	**0.2230**
		0.8	0.3161	0.3230	0.3824	0.3169	0.3275	0.3977	**0.3153**
		0.9	0.5980	0.6060	0.7214	0.5988	0.6353	0.8279	**0.5975**
	0.4	0.7	0.1691	0.1842	0.3446	0.1702	0.1727	0.2063	**0.1690**
		0.8	0.2388	0.2526	0.3913	0.2411	0.2460	0.3083	**0.2383**
		0.9	0.4486	0.4512	0.4618	0.4345	0.4732	0.6562	**0.4325**
	0.8	0.7	0.2151	0.2138	0.4132	0.2150	0.2215	0.2660	**0.2145**
		0.8	0.3034	0.3315	0.3054	0.3095	0.3158	0.3974	**0.3026**
		0.9	0.5680	0.5841	0.6969	0.5805	0.6078	0.8406	**0.5658**
1.05	-0.8	0.7	2.5389	2.6130	2.3181	2.7674	2.4132	**2.2618**	2.5237
		0.8	3.4890	3.6070	3.0855	3.8704	3.2611	**2.9630**	3.4503
		0.9	6.3131	6.6623	5.4033	7.1689	5.7154	**4.8778**	6.1244
	-0.4	0.7	0.6067	0.6175	0.7891	0.6170	0.6269	0.7678	**0.6058**
		0.8	0.8184	0.8368	0.9204	0.8460	0.8472	1.0678	**0.8101**
		0.9	1.4220	1.4600	1.5296	1.5137	1.4703	1.9508	**1.4217**
	0.0	0.7	0.2927	0.2951	0.3285	**0.2915**	0.3034	0.3583	0.2936
		0.8	0.3889	0.4005	0.4854	**0.3880**	0.4091	0.5084	0.3912
		0.9	0.6811	0.6991	0.8626	**0.6801**	0.7414	1.0039	0.6951
	0.4	0.7	0.2370	0.2581	0.4482	**0.2368**	0.2444	0.2916	0.2375
		0.8	0.3148	0.3339	0.5020	**0.3147**	0.3300	0.4209	0.3163
		0.9	0.5589	0.5685	0.5758	**0.5414**	0.6101	0.8797	0.5695
	0.8	0.7	0.2836	0.2857	0.4900	**0.2821**	0.2940	0.3525	0.2844
		0.8	0.3879	0.4025	0.6470	**0.3856**	0.4102	0.5273	0.3903
		0.9	0.7333	0.7441	0.9406	**0.7243**	0.8104	1.1643	0.7505
1.10	-0.8	0.7	2.7992	2.8805	2.5847	3.0395	2.6726	**2.5364**	2.7837
		0.8	3.7596	3.8891	3.3473	4.1684	3.5196	**3.2175**	3.7185
		0.9	6.6679	7.0329	5.6801	7.5857	6.0023	**5.0179**	6.4584
	-0.4	0.7	0.8424	0.8642	1.0722	0.8511	0.8768	1.0647	**0.8420**
		0.8	1.0637	1.0907	1.2133	1.0922	1.1135	1.4100	**1.0631**
		0.9	1.6834	1.7358	1.8510	1.7895	1.7792	2.4597	**1.6830**

	0.0	0.7	0.4998	0.5009	0.5347	**0.4965**	0.5194	0.6084	0.5014
		0.8	0.6053	0.6256	0.7619	**0.6020**	0.6436	0.8099	0.6099
		0.9	0.9269	0.9627	1.2238	**0.9224**	1.0442	1.4931	0.9552
	0.4	0.7	0.4388	0.4683	0.7064	**0.4376**	0.4534	0.5314	0.4400
		0.8	0.5381	0.5649	0.7839	**0.5344**	0.5697	0.7280	0.5416
		0.9	0.8712	0.8854	0.8810	**0.8578**	0.9796	1.4555	0.8955
	0.8	0.7	0.4855	0.4877	0.6868	**0.4811**	0.5039	0.5960	0.4870
		0.8	0.6279	0.6313	13.5673	**0.6146**	0.6710	0.8689	0.6330
		0.9	1.1701	1.1870	1.5068	**1.1453**	1.3171	1.9032	1.2044

Table 3. Restriction Rype 2, under $AR(1)$

τ	ρ	γ	$\hat{\beta}_{RS}$	$\hat{\beta}_R(k)$			$\hat{\beta}_{RS}(k)$		
			---------	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$
1	-0.8	0.7	0.6980	0.7243	0.5694	0.8706	0.6976	**0.5282**	0.6776
		0.8	0.9366	0.9796	0.7144	1.1934	0.9036	**0.7845**	0.8865
		0.9	1.6301	1.7253	1.1006	2.0467	1.4916	1.4501	**1.4417**
	-0.4	0.7	0.3118	0.3127	0.3368	0.3310	0.4033	0.4763	**0.3116**
		0.8	0.4246	0.4217	0.4218	0.4585	0.6318	**0.4097**	0.4543
		0.9	0.7275	0.7118	0.6011	0.7937	0.7841	**0.5921**	1.0185
	0.0	0.7	0.1548	0.1511	0.2027	**0.1415**	0.2066	0.4572	0.1620
		0.8	0.2302	0.2129	0.2732	**0.2016**	0.3843	0.5400	0.2972
		0.9	0.4991	0.3730	0.4167	0.3729	**0.3497**	0.3917	0.4990
	0.4	0.7	0.0883	**0.0849**	0.1045	0.0889	0.0947	0.1535	0.0886
		0.8	0.1245	**0.1178**	0.1463	0.1240	0.1402	0.2580	0.1261
		0.9	0.2358	**0.2107**	0.2517	0.2216	0.3449	0.2493	0.2718
	0.8	0.7	0.0117	0.0114	0.0168	**0.0113**	0.0128	0.0219	0.0118
		0.8	0.0167	0.0161	0.0257	**0.0160**	0.0190	0.0363	0.0170
		0.9	0.0324	0.0305	0.0554	**0.0297**	0.0416	0.0975	0.0354
1.05	-0.8	0.7	0.7754	0.8045	**0.6640**	0.9482	0.7906	0.7019	0.7564
		0.8	1.0096	1.0566	**0.8052**	1.2705	0.9872	1.3303	0.9601
		0.9	1.6891	1.7914	**1.1781**	2.1176	1.5473	1.4846	1.4964
	-0.4	0.7	0.3997	0.3993	0.4419	0.4093	0.5309	**0.3972**	0.4152
		0.8	0.5150	**0.5094**	0.5296	0.5384	0.7954	0.8334	0.5579
		0.9	0.8219	0.7952	0.7063	0.8729	0.6451	**0.6269**	0.7027
	0.0	0.7	0.2400	0.2332	0.3018	**0.2169**	0.4688	0.8696	0.2577
		0.8	0.3376	0.3025	0.3868	0.2808	**0.2485**	0.2684	0.2674
		0.9	0.7255	0.4730	0.5472	0.4627	0.4621	**0.4524**	0.4951
	0.4	0.7	0.1671	**0.1630**	0.1964	0.1639	0.1815	0.2697	0.1685
		0.8	0.2181	0.2082	0.2649	**0.2071**	0.2591	0.4660	0.2251
		0.9	0.4013	0.3456	0.4370	**0.3423**	0.8490	0.3838	0.5537
	0.8	0.7	0.0927	**0.0919**	0.1024	0.0914	0.0952	0.1124	0.0930
		0.8	0.1172	0.1145	0.1368	**0.1133**	0.1236	0.1627	0.1182
		0.9	0.2089	0.1946	0.2521	**0.1927**	0.2359	0.3866	0.2177

1.10	-0.8	0.7	1.0057	1.0391	**0.9129**	1.1833	1.0453	1.0953	0.9888
		0.8	1.2332	1.2867	**1.0487**	1.5060	1.2428	1.7984	1.1870
		0.9	1.8854	2.0003	1.3948	2.3402	1.8226	**1.3883**	1.7120
	-0.4	0.7	0.6446	**0.6411**	0.7060	0.6424	0.8542	0.8900	0.6701
		0.8	0.7676	0.7533	0.7979	0.7749	0.6542	**0.6358**	0.8410
		0.9	1.0988	1.0311	**0.9684**	1.1072	1.0602	1.0134	1.3054
	0.0	0.7	0.4851	0.4690	0.5652	**0.4416**	0.5961	0.5883	0.5524
		0.8	0.6439	0.5519	0.6715	**0.5139**	0.6114	0.7704	0.6211
		0.9	1.8613	0.7405	0.8518	0.7173	**0.6040**	0.6633	0.6201
	0.4	0.7	0.3992	0.3926	0.4547	**0.3883**	0.4340	0.6020	0.4034
		0.8	0.4883	0.4670	0.5715	**0.4541**	0.6028	0.6941	0.5102
		0.9	0.9298	0.6986	0.8261	0.6824	0.7024	**0.6311**	0.6391
	0.8	0.7	0.3350	0.3314	0.3581	**0.3310**	0.3417	0.3897	0.3356
		0.8	0.4137	**0.4000**	0.4510	0.4022	0.4296	0.5476	0.4152
		0.9	0.6840	0.6206	0.6906	0.6435	0.7623	**0.6173**	0.7005

Table 4. Restriction Rype 2, under $MA(1)$

τ	ρ	γ	$\hat{\beta}_{RS}$	$\hat{\beta}_R(k)$			$\hat{\beta}_{RS}(k)$		
			---------	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$	$\hat{k}_{HK}$	$\hat{k}_{HKB}$	$\hat{k}_{LW}$
1	-0.8	0.7	1.4404	1.5400	**1.1412**	1.7738	1.2782	1.1798	1.4035
		0.8	1.9953	2.1391	**1.5098**	2.4668	1.7543	1.6201	1.9198
		0.9	3.6677	3.9326	**2.5758**	4.4163	3.2625	3.4312	3.4412
	-0.4	0.7	0.3120	**0.3076**	0.3580	0.3146	0.4381	0.3197	0.3288
		0.8	0.4305	**0.4141**	0.4435	0.4371	0.7705	0.4774	0.4872
		0.9	0.7512	0.6889	0.6130	0.7571	0.6185	**0.6111**	0.6972
	0.0	0.7	0.1548	0.1511	0.2027	**0.1415**	0.2066	0.4572	0.1620
		0.8	0.2302	0.2129	0.2732	**0.2016**	0.8843	0.5400	0.2972
		0.9	0.4991	0.3730	0.4167	0.3729	**0.2497**	0.3176	0.4991
	0.4	0.7	0.1150	**0.1099**	0.1419	0.1121	0.1329	0.2366	0.1169
		0.8	0.1650	**0.1533**	0.1965	0.1570	0.2082	0.4327	0.1724
		0.9	0.3371	**0.2742**	0.3226	0.2838	0.2744	0.3999	0.3465
	0.8	0.7	0.1604	**0.1475**	0.1536	0.1516	0.1689	0.2181	0.1613
		0.8	0.2358	**0.2057**	0.2111	0.2129	0.4271	0.2642	0.2612
		0.9	0.5401	0.3670	**0.3492**	0.3849	0.4467	0.5196	0.5890
1.05	-0.8	0.7	1.5503	1.6467	1.2600	1.8744	1.3635	**1.1427**	1.5110
		0.8	2.1080	2.2497	1.6230	2.5761	1.7846	**1.3715**	2.0156
		0.9	3.7936	4.0711	2.6971	4.5609	3.0331	**2.0406**	3.4069
	-0.4	0.7	0.4125	0.4040	0.4748	**0.3994**	0.5861	0.4462	0.4371
		0.8	0.5372	**0.5136**	0.5615	0.5266	0.9355	0.6584	0.6094
		0.9	0.8646	0.7890	0.7273	0.8534	**0.6527**	0.6681	0.6539
	0.0	0.7	0.2400	0.2332	0.3018	**0.2169**	0.4688	0.8696	0.2577
		0.8	0.3376	0.3025	0.3868	0.2808	**0.2476**	0.4524	0.2616
		0.9	0.7255	0.4730	0.5472	0.4627	**0.2050**	0.3249	0.4501

	0.4	0.7	0.1968	0.1892	0.2387	**0.1864**	0.2286	0.3790	0.2008
		0.8	0.2698	0.2473	0.3258	**0.2391**	0.3707	0.8323	0.2893
		0.9	0.5969	0.4147	0.5187	0.4046	**0.2431**	0.3809	0.4871
	0.8	0.7	0.2367	**0.2213**	0.2322	0.2245	0.2507	0.3231	0.2385
		0.8	0.3569	0.2969	0.3247	0.2971	0.3683	**0.2586**	0.2901
		0.9	1.1817	0.5206	0.5419	0.5297	**0.3902**	0.6215	0.7157
1.10	-0.8	0.7	1.8582	1.9501	**1.5572**	2.1812	1.6444	1.3696	1.8141
		0.8	2.4506	2.5723	1.9287	2.9080	2.0842	**1.5978**	2.3460
		0.9	4.2842	4.4860	3.0600	4.9966	3.4364	**2.3021**	3.8523
	-0.4	0.7	0.6767	0.6643	0.7530	**0.6496**	0.9348	0.7238	0.7125
		0.8	0.8120	0.7813	0.8427	0.7884	0.7515	**0.5341**	0.9135
		0.9	1.1604	1.0636	**1.0051**	1.1323	1.0793	1.0949	1.1062
	0.0	0.7	0.4851	0.4690	0.5652	**0.4416**	0.5961	0.5883	0.5524
		0.8	0.6439	0.5519	0.6715	0.5139	**0.5114**	0.7704	0.6241
		0.9	1.8613	0.7405	0.8518	0.7173	**0.6040**	0.6633	0.6545
	0.4	0.7	0.4339	0.4196	0.5012	**0.4085**	0.5013	0.7794	0.4431
		0.8	0.5649	0.5090	0.6394	0.4812	**0.4085**	0.4910	0.6416
		0.9	1.4526	0.7634	0.9046	**0.7342**	1.1244	1.0790	1.0242
	0.8	0.7	0.4632	**0.4412**	0.4624	0.4431	0.4968	0.4746	0.4677
		0.8	0.7064	0.5546	0.6113	**0.5459**	0.5829	0.5576	0.5948
		0.9	1.1255	**0.9156**	0.9353	0.9329	0.9326	0.9213	0.9255