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HAAR WAVELET METHOD FOR THE SOLUTION OF TWO-DIMENSIONAL PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Parabolic partial differential equations arise in the mathematical modelling of many physical phenomena. In this paper, we use Haar wavelet method for the numerical solution of two-dimensional heat equation. The basic idea of Haar wavelet collocation method is to convert the partial differential equation into a system of algebraic equations that involves a finite number of variables. The numerical results are compared with the exact solution to prove the accuracy of the Haar wavelet method.

Keywords: Parabolic partial differential equations, two-dimensional heat equation, Haar wavelets, collocation points.

Mathematics Subject Classification: 65T60

1 Introduction

Parabolic partial differential equations (PDEs) are used to describe variety of problems in science including heat diffusion, ocean acoustic propagation, physical or mathematical systems with a time variable, and processes that behave essentially like heat diffusing through a solid. Analytical methods like method of separation of variables [1] and differential transform method [2] have been widely used to solve parabolic PDEs. Semi-analytical methods like Adomian decomposition method [3, 4] and homotopy analysis method [5] have been applied to solve parabolic PDEs. Numerical methods like finite difference methods [6, 7], finite volume methods [8, 9], explicit Runge-Kutta methods [10], method of lines [11], interpolation technique [12], methods using radial basis functions [13] and Pade approximation [14] have also been used to solve parabolic PDEs.

In the recent years, wavelets have been widely used to solve differential equations. Alfred Haar, a Hungarian mathematician introduced Haar wavelets in 1910. The Haar wavelets consist of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. An advantage of these wavelets is the possibility to integrate them analytically arbitrary times. They are the simplest possible wavelets and are often known as a first order Daubechies wavelet which are conceptually simple, fast, memory efficient and exactly reversible. Sumana and Achala [15] have given a brief report on Haar wavelets.

Lepik [16] applied the Haar wavelet method along with the segmentation technique to solve differential equations. Khalid et. al. [17] solved Airy differential equation using Haar wavelets. Shi and Cao [18] applied Haar wavelets to solve eigenvalue problems of high order differential equations. Lepik [19] applied Haar wavelets to solve evolution equations. Bujurke et. al. [20] applied wavelet-multigrid method to solve elliptic partial differential equations. Lepik [21] used two-dimensional Haar wavelets to solve diffusion equation and Poisson equation.

The paper is organized as follows. The Haar wavelet preliminaries and the function approximation are presented in Section 2 and Section 3 respectively. The method of solution of the two-dimensional heat equation using Haar wavelets is proposed in Section 4. The numerical examples and discussions are presented in Section 5. The conclusions drawn are presented in Section 6.

2 Preliminaries of Haar Wavelets

The Haar wavelet family for $x \in [0,1]$ is defined [22] as follows

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1, \xi_2) \\ -1 & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

where

$$\xi_1 = \frac{k}{m},\ \xi_2 = \frac{k+0.5}{m},\ \xi_3 = \frac{k+1}{m}. \tag{2}$$

Here $m = 2^n$, $n = 0,1,\ldots,J$ indicates the level of the wavelet; $k = 0,1,\ldots,m-1$ is the translation parameter. J is the maximum level of resolution. The index i in equation (1) is calculated by the formula $i = m+k+1$. In the case of minimum values $m=1, k=0$ we have $i=2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i=1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

We require the following integrals in order to solve second order partial differential equations.

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} x-\xi_1 & \text{for } x \in [\xi_1, \xi_2) \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \frac{1}{2}(x-\xi_1)^2 & \text{for } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3) \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3, 1] \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

3 Function Approximation

According to the two-dimensional multi-resolution analysis, any function $f(x,y)$ which is square integrable on $[0,1)\times[0,1)$ can be expressed in terms of two-dimensional Haar series as follows.

$$f(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a(i,j)\, h_i(x)\, h_j(y). \tag{6}$$

Here, the expansion of $f(x,y)$ is an infinite series. If $f(x,y)$ is approximated as piecewise constant in each sub-area, then it will be terminated at finite terms, that is,

$$f(x,y) = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a(i,j)\, h_i(x)\, h_j(y), \tag{7}$$

where the wavelet coefficients $a(i,j), i = 1,2,\ldots,2M_1,\ j = 1,2,\ldots,2M_2$ are to be determined.

4 Method of Solution

Consider the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\ 0 \le x, y \le 1, t \ge 0, \tag{8}$$

with initial and boundary conditions

$$u(x, y, 0) = F(x, y), 0 \le x, y \le 1, \tag{9}$$

$$\left.\begin{aligned} u(x,0,t) &= f_1(x,t) \\ u(x,1,t) &= f_2(x,t) \end{aligned}\right\} 0 \le x \le 1, t \ge 0, \tag{10}$$

$$\left.\begin{aligned} u(0,y,t) &= g_1(y,t) \\ u(1,y,t) &= g_2(y,t) \end{aligned}\right\} 0 \le y \le 1, t \ge 0. \tag{11}$$

Let us divide the interval [0,1] into N equal parts of length $\Delta t = \frac{T}{N}$ and denote $t_s = (s-1)\Delta t,\ s = 1, 2, 3 ... N.$

Let the Haar wavelet solution be in the form

$$\dot{u}_{xxyy}(x, y) = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i, j) h_i(x) h_j(y) \tag{12}$$

Integrating (13) w.r.t. t in the limits $[t_s, t]$, we have

$$u_{xxyy}(x, y, t) = (t - t_s)\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i, j) h_i(x) h_j(y) + u_{xxyy}(x, y, t_s) \tag{13}$$

Integrating equation (13) twice w.r.t. y in the limits $[0, y]$ and using (10) gives

$$\begin{aligned} u_{xx}(x, y, t) = \quad & (t - t_s)\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i, j) h_i(x)[q_j(y) - y q_j(1)] + u_{xx}(x, y, t_s) \\ & + y[f_2''(x,t) - f_2''(x,t_s)] + (1-y)[f_1''(x,t) - f_1''(x,t_s)] \end{aligned} \tag{14}$$

Integrating equation (13) twice w.r.t. x in the limits $[0, x]$ and using (11) leads to

$$\begin{aligned} u_{yy}(x, y, t) = \quad & (t - t_s)\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i, j)[q_i(x) - x q_i(1)] h_j(y) + u_{yy}(x, y, t_s) \\ & + x[g_2''(y,t) - g_2''(y,t_s)] + (1-x)[g_1''(y,t) - g_1''(y,t_s)] \end{aligned} \tag{15}$$

Integrating equation (14) twice w.r.t x in the limits $[0, x]$ and using (11), we arrive at

$$
\begin{aligned}
u(x,y,t) = \; & (t-t_s)\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i,j)[q_i(x)-xq_i(1)][q_j(y)-yq_j(1)]+u(x,y,t_s) \\
& +x[g_2(y,t)-g_2(y,t_s)]+(1-x)[g_1(y,t)-g_1(y,t_s)]+y[f_2(x,t) \\
& -f_2(x,t_s)]+(1-y)[f_1(x,t)-f_1(x,t_s)]-xy[f_2(1,t)-f_2(1,t_s)] \\
& -x(1-y)[f_1(1,t)-f_1(1,t_s)]-(1-x)y[f_2(0,t)-f_2(0,t_s)] \\
& -(1-x)(1-y)[f_1(0,t)-f_1(0,t_s)]
\end{aligned} \tag{16}
$$

Differentiating equation (16) w.r.t t gives

$$
\begin{aligned}
\dot{u}(x,y,t) = \; & \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i,j)[q_i(x)-xq_i(1)][q_j(y)-yq_j(1)]+x\,\dot{g}_2(y,t) \\
& +(1-x)\dot{g}_1(y,t)+y\,\dot{f}_2(x,t)+(1-y)\,\dot{f}_1(x,t)-xy\,\dot{f}_2(1,t) \\
& -x(1-y)\,\dot{f}_1(1,t)-(1-x)y\dot{f}_2(0,t)-(1-x)(1-y)\,\dot{f}_1(0,t)
\end{aligned} \tag{17}
$$

The wavelet collocation points are defined as

$$
x_l = \frac{l-0.5}{2M_1}, \quad l = 1,2,\ldots,2M_1 \tag{18}
$$

$$
y_n = \frac{n-0.5}{2M_2}, \quad n = 1,2,\ldots,2M_2 \tag{19}
$$

Substituting equations (14), (15) and (17) in equation (8), and taking $x \to x_l$. $y \to y_n$ and $t \to t_{s+1}$ in the resultant equation and equations (14)-(16), we obtain

$$
\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i,j)A_s(i,j,l,n) = \phi_s(x_l,y_n), \quad s = 0,1,2,\ldots \tag{20}
$$

where

$$
\begin{aligned}
A_s(i,j,l,n) = \; & [q_i(x_l)-x_lq_i(1)][q_j(y_n)-y_nq_j(1)]-c(\Delta t)h_i(x_l)[q_j(y_n)-y_nq_j(1)] \\
& -c(\Delta t)[q_i(x_l)-x_lq_i(1)]h_j(y_n)
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\phi_s(x_l,y_n) = \; & cu_{xx}(x_l,y_n,t_s)+cu_{yy}(x_l,y_n,t_s)+c\{y_n[f_2''(x_l,t_{s+1})-f_2''(x_l,t_s)] \\
& +(1-y_n)[f_1''(x_l,t_{s+1})-f_1''(x_l,t_s)]+x_l[g_2''(y_n,t_{s+1})-g_2''(y_n,t_s)] \\
& +(1-x_l)[g_1''(y_n,t_{s+1})-g_1''(y_n,t_s)]\}-x_l\dot{g}_2(y_n,t_{s+1}) \\
& -(1-x_l)\dot{g}_1(y_n,t_{s+1})-y_n\,\dot{f}_2(x_l,t_{s+1})-(1-y_n)\,\dot{f}_1(x_l,t_{s+1}) \\
& +x_ly_n\,\dot{f}_2(1,t_{s+1})+x_l(1-y_n)\,\dot{f}_1(1,t_{s+1})+(1-x_l)y_n\,\dot{f}_2(0,t_{s+1}) \\
& +(1-x_l)(1-y_n)\,\dot{f}_1(0,t_{s+1})
\end{aligned} \tag{22}
$$

$$\begin{aligned}u(x_l, y_n, t_{s+1}) = \quad & \Delta t\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i,j)[q_i(x_l) - x_l q_i(1)][q_j(y_n) - y_n q_j(1)] + u(x_l, y_n, t_s) \\ & + x_l[g_2(y_n, t_{s+1}) - g_2(y_n, t_s)] + (1 - x_l)[g_1(y_n, t_{s+1}) - g_1(y_n, t_s)] \\ & + y_n[f_2(x_l, t_{s+1}) - f_2(x_l, t_s)] + (1 - y_n)[f_1(x_l, t_{s+1}) - f_1(x_l, t_s)] \\ & - x_l y_n[f_2(1, t_{s+1}) - f_2(1, t_s)] - x_l(1 - y_n)[f_1(1, t_{s+1}) - f_1(1, t_s)] \\ & - (1 - x_l) y_n[f_2(0, t_{s+1}) - f_2(0, t_s)] - (1 - x_l)(1 - y_n)[f_1(0, t_{s+1}) \\ & - f_1(0, t_s)]\end{aligned} \tag{23}$$

$$\begin{aligned}u_{xx}(x_l, y_n, t_{s+1}) = \quad & \Delta t\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i,j) h_i(x_l)[q_j(y_n) - y_n q_j(1)] + u_{xx}(x_l, y_n, t_s) \\ & + y_n[f_2''(x_l, t_{s+1}) - f_2''(x_l, t_s)] + (1 - y_n)[f_1''(x_l, t_{s+1}) - f_1''(x_l, t_s)]\end{aligned} \tag{24}$$

$$\begin{aligned}u_{yy}(x_l, y_n, t_{s+1}) = \quad & \Delta t\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_s(i,j)[q_i(x_l) - x_l q_i(1)] h_j(y_n) + u_{yy}(x_l, y_n, t_s) \\ & + x_l[g_2''(y_n, t_{s+1}) - g_2''(y_n, t_s)] + (1 - x_l)[g_1''(y_n, t_{s+1}) - g_1''(y_n, t_s)]\end{aligned} \tag{25}$$

Using the initial conditions (9), we have

$$\begin{aligned}u(x_l, y_n, 0) &= F(x_l, y_n) \\ u_{xx}(x_l, y_n, 0) &= F_{xx}(x_l, y_n) \\ u_{yy}(x_l, y_n, 0) &= F_{yy}(x_l, y_n)\end{aligned} \tag{26}$$

The wavelet coefficients $a_s(i, j),\ i = 1, 2, \ldots, 2M_1,\ j = 1, 2, \ldots, 2M_2$ can be successively calculated from equation (20). This process is started with equation (26). These coefficients are then substituted in equations (23)-(25) to obtain the approximate solutions at different time levels.

5 Numerical Examples and Discussion

In this section, examples are considered to check the efficiency and accuracy of the Haar wavelet collocation method (HWCM). Lagrange bivariate interpolation is used to find the solution at the specified points. The entire computational work has been done with the help of MATLAB software.

Example 1:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\quad 0\le x,y\le 1,t\ge 0,$$
$$u(x,y,0)=\cos(\pi(x-y))-\cos(\pi(x+y)),0\le x,y\le 1,$$
$$\left.\begin{aligned}u(x,0,t)=0\\u(x,1,t)=0\end{aligned}\right\}0\le x\le 1,t\ge 0,\tag{27}$$
$$\left.\begin{aligned}u(0,y,t)=0\\u(1,y,t)=0\end{aligned}\right\}0\le y\le 1,t\ge 0.$$

The exact solution is

$$u(x,y,t)=2\sin(\pi x)\sin(\pi y)e^{-2\pi^2 t}\tag{28}$$

The HWCM solution of the example at $t=0.01,0.05,0.1$ with $M_1=16$, $M_2=16$ and $\Delta t=0.0001$ in Tables 1,2,3. The results are compared with the exact solution. Figures 1,2 show the comparison of the HWCM solution with the exact solution and the physical behavior of the HWCM solution in contour and 3D at $t=0.1$. If $u_{ex}(x,y,t_s)$ is the exact solution (28) at $t=t_s$, we define the error estimate as

$$\nu(t_s)=\frac{1}{2M_1 2M_2}\left\|u(x,y,t_s)-u_{ex}(x,y,t_s)\right\|\tag{29}$$

We have obtained the following error estimates for $M_1=16$, $M_2=16$ and $\Delta t=0.0001$.

1. $\nu(0.01)=5.2075E-06$ in L_2 space and $\nu(0.01)=6.6250E-06$ in L_∞ space.
2. $\nu(0.05)=1.4590E-05$ in L_2 space and $\nu(0.05)=1.8561E-05$ in L_∞ space.
3. $\nu(0.1)=1.5810E-05$ in L_2 space and $\nu(0.1)=2.0114E-05$ in L_∞ space.

Example 2:

$$\frac{\partial u}{\partial t}=\frac{1}{2\pi^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\quad 0\le x,y\le 1,t\ge 0,$$
$$u(x,y,0)=2\sin(\pi(x-y))\sin(\pi(x+y)),0\le x,y\le 1,$$
$$\left.\begin{aligned}u(x,0,t)=0\\u(x,1,t)=0\end{aligned}\right\}0\le x\le 1,t\ge 0,\tag{30}$$
$$\left.\begin{aligned}u(0,y,t)=0\\u(1,y,t)=0\end{aligned}\right\}0\le y\le 1,t\ge 0.$$

The exact solution is

$$u(x,y,t)=e^{-t}\cos\left(\pi\left(x-\frac{1}{2}\right)\right)-\cos\left(\pi\left(y-\frac{1}{2}\right)\right)\tag{31}$$

The HWCM solution of the example at $t=0.01,0.05,0.1$ with $M_1=16$, $M_2=16$ and

$\Delta t = 0.0001$ in Tables 4,5,6. The results are compared with the exact solution. Figures 3,4 show the comparison of the HWCM solution with the exact solution and the physical behavior of the HWCM solution in contour and 3D at $t = 0.1$. We have obtained the following error estimates for $M_1 = 16$, $M_2 = 16$ and $\Delta t = 0.0001$.

1. $\nu(0.01) = 2.9748E-07$ in L_2 space and $\nu(0.01) = 3.7846E-07$ in L_∞ space.
2. $\nu(0.05) = 1.4352E-06$ in L_2 space and $\nu(0.05) = 1.8259E-06$ in L_∞ space.
3. $\nu(0.1) = 7.2213E-04$ in L_2 space and $\nu(0.1) = 9.1871E-04$ in L_∞ space.

Example 3:

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, 0 \le x, y \le 1, t \ge 0, \\ u(x, y, 0) = (1-y)e^x,\ 0 \le x, y \le 1 \\ \left.\begin{aligned} u(x,0,t) &= e^{x+t} \\ u(x,1,t) &= 0 \end{aligned}\right\} 0 \le x \le 1, t \ge 0, \\ \left.\begin{aligned} u(0,y,t) &= (1-y)e^t \\ u(1,y,t) &= (1-y)e^{1+t} \end{aligned}\right\} 0 \le y \le 1, t \ge 0. \end{gathered} \tag{32}$$

The exact solution is

$$u(x, y, t) = (1-y)e^{x+t} \tag{33}$$

The HWCM solution of the example at $t = 0.01, 0.05, 0.1$ with $M_1 = 4$, $M_2 = 4$ and $\Delta t = 0.001$ in Tables 7,8,9. The results are compared with the exact solution. Figures 5,6 show the comparison of the HWCM solution with the exact solution and the physical behavior of the HWCM solution in contour and 3D at $t = 0.1$. We have obtained the following error estimates for $M_1 = 4$, $M_2 = 4$ and $\Delta t = 0.001$.

1. $\nu(0.01) = 1.8815E-17$ in L_2 space and $\nu(0.01) = 3.0358E-17$ in L_∞ space.
2. $\nu(0.05) = 1.0148E-16$ in L_2 space and $\nu(0.05) = 1.4615E-16$ in L_∞ space.
3. $\nu(0.1) = 1.0614E-16$ in L_2 space and $\nu(0.1) = 1.2273E-16$ in L_∞ space.

6 Conclusion

In this paper, an efficient numerical scheme based on uniform Haar wavelets is used to solve parabolic partial differential equation, namely, two-dimensional heat equation. The numerical scheme is tested for three examples. The obtained numerical results are compared with the exact solutions. We observe that the error estimates are negligibly small in the case of nonlocal boundary conditions (i.e., Example 3) for a small number of grid points. Thus the Haar wavelet method guarantees the necessary accuracy with a small number of grid points and a wide class of PDEs can be solved using this approach. This method takes care of boundary conditions automatically and hence it is the most convenient method for solving boundary value problems. This method can also be used to solve nonlinear PDEs.

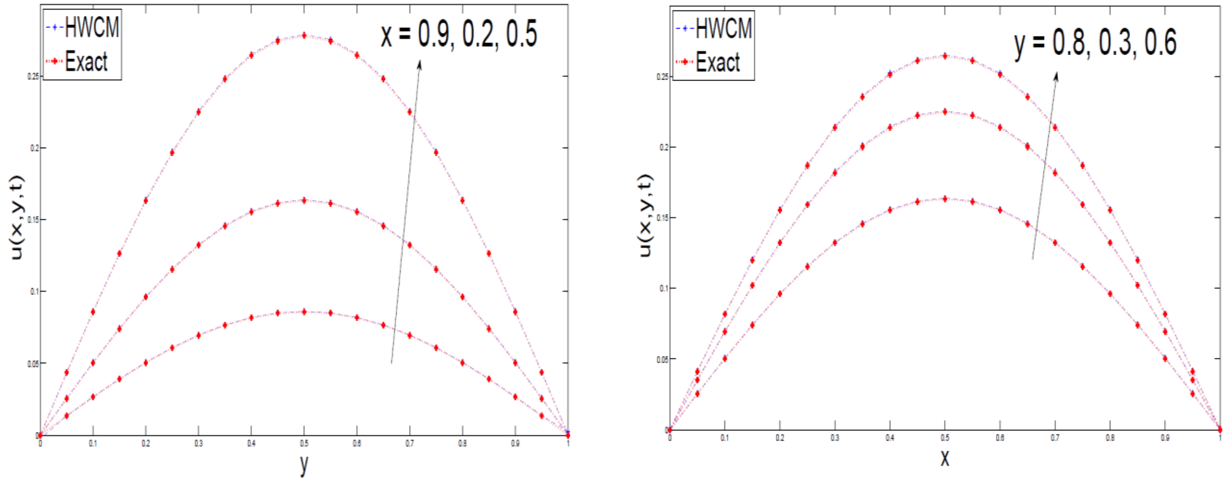

Figure 1: Comparison of the HWCM solution and exact solution of Example 1 at $t = 0.1$

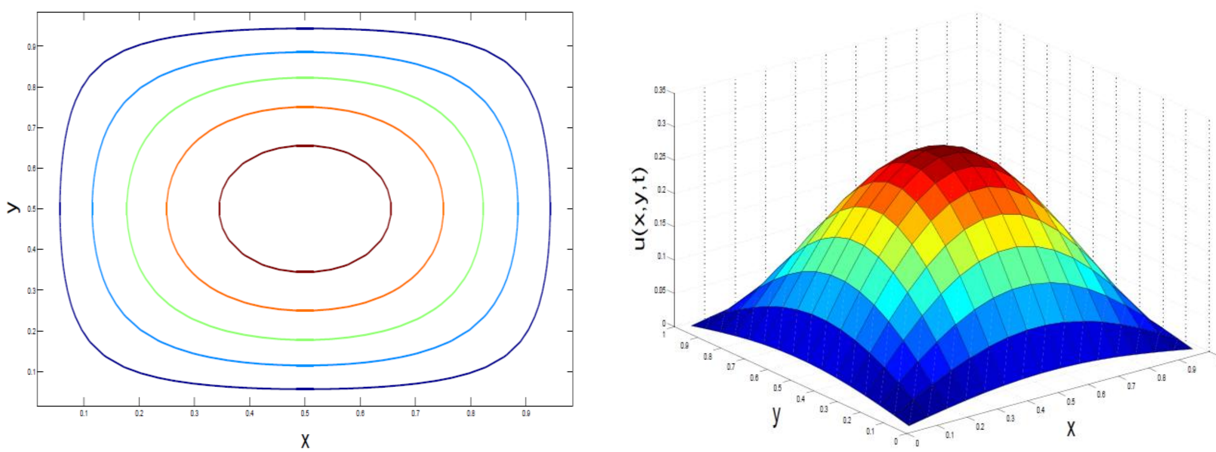

Figure 2: Physical behaviour of the HWCM solution of Example 1 at $t = 0.1$

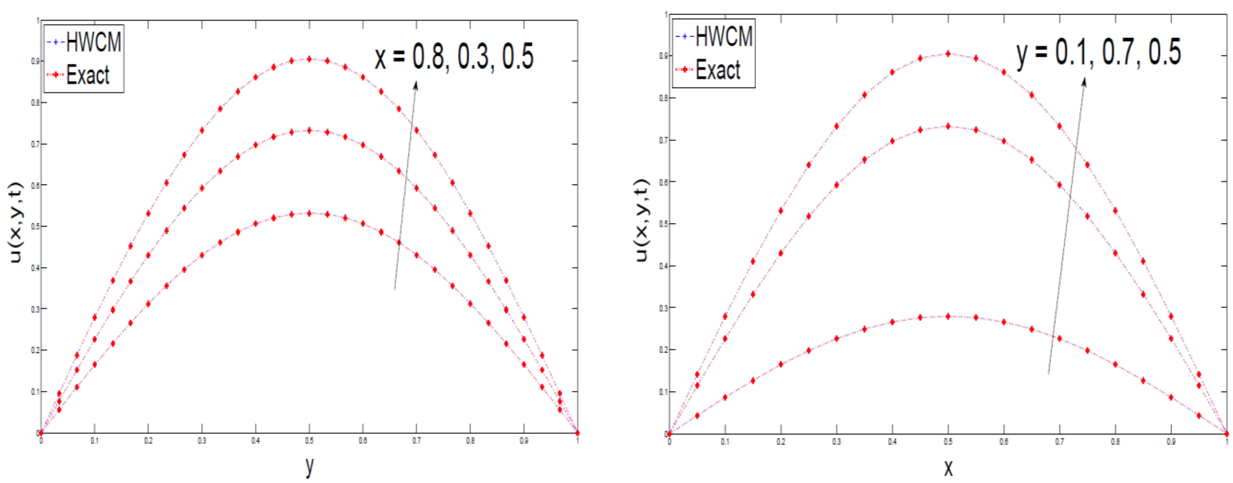

Figure 3: Comparison of the HWCM solution and exact solution of Example 2 at $t = 0.1$

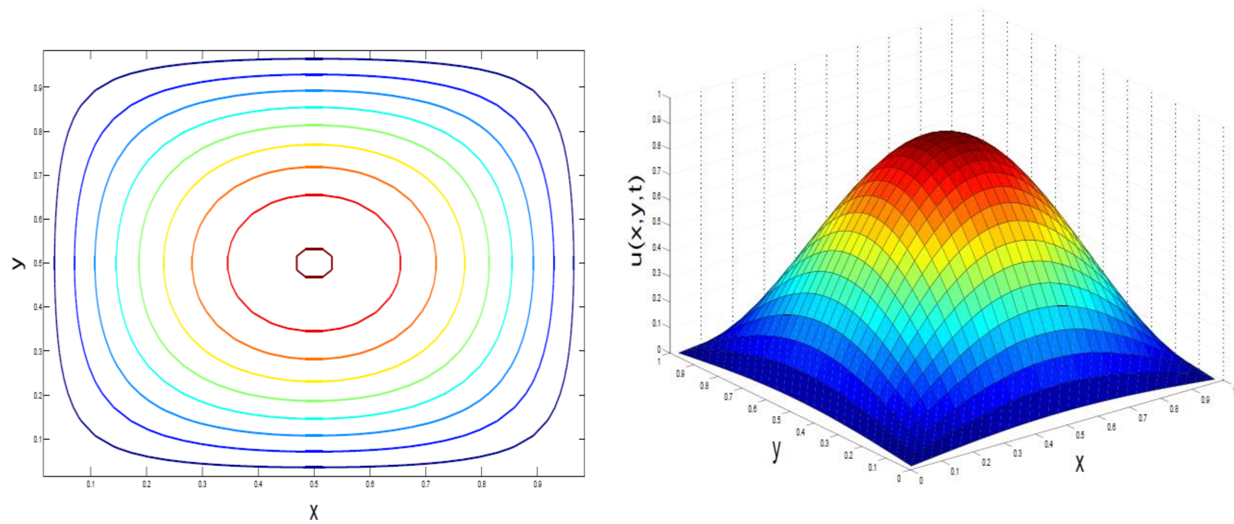

Figure 4: Physical behaviour of the HWCM solution of Example 2 at $t = 0.1$

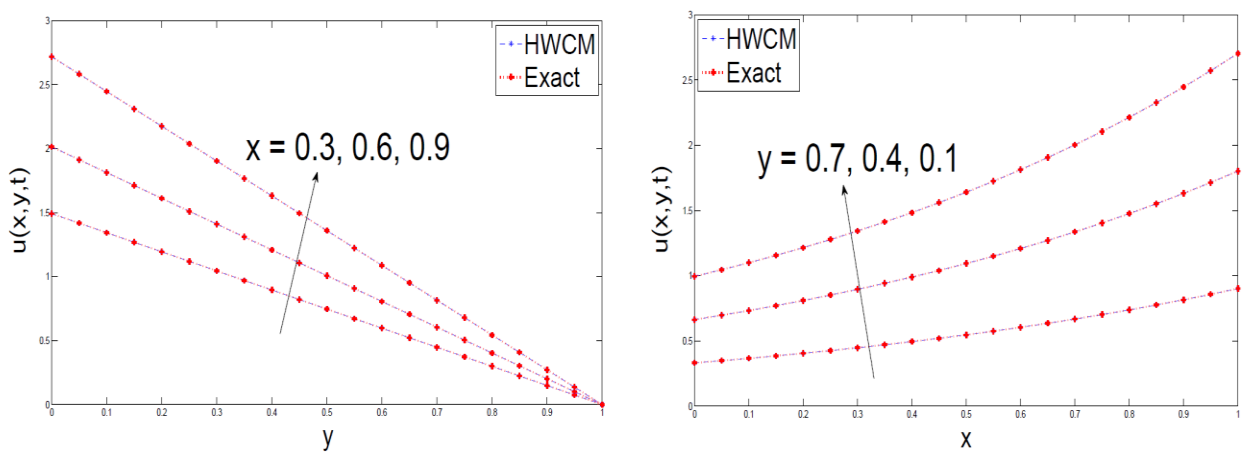

Figure 5: Comparison of the HWCM solution and exact solution of Example 3 at $t = 0.1$

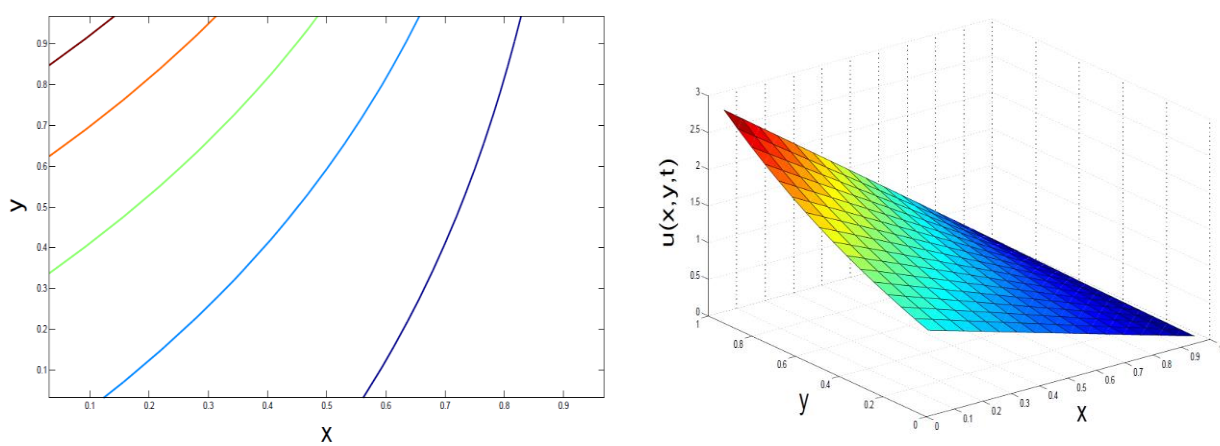

Figure 6: Physical behaviour of the HWCM solution of Example 3 at $t = 0.1$

Table 1: Comparison of HWCM solution and exact solution of Example 1 at $t = 0.01$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.29825855	0.29819802	(0.5,0.6)	1.56170205	1.56138509
(0.1,0.4)	0.48259247	0.48249453	(0.5,0.8)	0.96518495	0.96498905
(0.1,0.6)	0.48259247	0.48249453	(0.7,0.2)	0.78085103	0.78069254
(0.1,0.8)	0.29825855	0.29819802	(0.7,0.4)	1.26344350	1.26318707
(0.3,0.2)	0.78085103	0.78069254	(0.7,0.6)	1.26344350	1.26318707
(0.3,0.4)	1.26344350	1.26318707	(0.7,0.8)	0.78085103	0.78069254
(0.3,0.6)	1.26344350	1.26318707	(0.9,0.2)	0.29825855	0.29819802
(0.3,0.8)	0.78085103	0.78069254	(0.9,0.4)	0.48259247	0.48249453
(0.5,0.2)	0.96518495	0.96498905	(0.9,0.6)	0.48259247	0.48249453
(0.5,0.4)	1.56170205	1.56138509	(0.9,0.8)	0.29825855	0.29819802

Table 2: Comparison of HWCM solution and exact solution of Example 1 at $t = 0.05$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.13556365	0.13539405	(0.5,0.6)	0.70982048	0.70893244
(0.1,0.4)	0.21934659	0.21907217	(0.5,0.8)	0.43869318	0.43814434
(0.1,0.6)	0.21934659	0.21907217	(0.7,0.2)	0.35491024	0.35446622
(0.1,0.8)	0.13556365	0.13539405	(0.7,0.4)	0.57425683	0.57353839
(0.3,0.2)	0.35491024	0.35446622	(0.7,0.6)	0.57425683	0.57353839
(0.3,0.4)	0.57425683	0.57353839	(0.7,0.8)	0.35491024	0.35446622
(0.3,0.6)	0.57425683	0.57353839	(0.9,0.2)	0.13556365	0.13539405
(0.3,0.8)	0.35491024	0.35446622	(0.9,0.4)	0.21934659	0.21907217
(0.5,0.2)	0.43869318	0.43814434	(0.9,0.6)	0.21934659	0.21907217
(0.5,0.4)	0.70982048	0.70893244	(0.9,0.8)	0.13556365	0.13539405

Table 3: Comparison of HWCM solution and exact solution of Example 1 at $t = 0.1$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.05064621	0.05046242	(0.5,0.6)	0.26518702	0.26422468
(0.1,0.4)	0.08194730	0.08164992	(0.5,0.8)	0.16389459	0.16329983
(0.1,0.6)	0.08194730	0.08164992	(0.7,0.2)	0.13259351	0.13211234
(0.1,0.8)	0.05064621	0.05046242	(0.7,0.4)	0.21454081	0.21376225
(0.3,0.2)	0.13259351	0.13211234	(0.7,0.6)	0.21454081	0.21376225
(0.3,0.4)	0.21454081	0.21376225	(0.7,0.8)	0.13259351	0.13211234
(0.3,0.6)	0.21454081	0.21376225	(0.9,0.2)	0.05064621	0.05046242
(0.3,0.8)	0.13259351	0.13211234	(0.9,0.4)	0.08194730	0.08164992
(0.5,0.2)	0.16389459	0.16329983	(0.9,0.6)	0.08194730	0.08164992
(0.5,0.4)	0.26518702	0.26422468	(0.9,0.8)	0.05064621	0.05046242

Table 4: Comparison of HWCM solution and exact solution of Example 2 at $t = 0.01$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.17983179	0.17982833	(0.5,0.6)	0.94161145	0.94159335
(0.1,0.4)	0.29097394	0.29096835	(0.5,0.8)	0.58194788	0.58193669
(0.1,0.6)	0.29097394	0.29096835	(0.7,0.2)	0.47080573	0.47079667
(0.1,0.8)	0.17983179	0.17982833	(0.7,0.4)	0.76177967	0.76176502
(0.3,0.2)	0.47080573	0.47079667	(0.7,0.6)	0.76177967	0.76176502
(0.3,0.4)	0.76177967	0.76176502	(0.7,0.8)	0.47080573	0.47079667
(0.3,0.6)	0.76177967	0.76176502	(0.9,0.2)	0.17983179	0.17982833
(0.3,0.8)	0.47080573	0.47079667	(0.9,0.4)	0.29097394	0.29096835
(0.5,0.2)	0.58194788	0.58193669	(0.9,0.6)	0.29097394	0.29096835
(0.5,0.4)	0.94161145	0.94159335	(0.9,0.8)	0.17983179	0.17982833

Table 5: Comparison of HWCM solution and exact solution of Example 2 at $t = 0.05$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.17279384	0.17277716	(0.5,0.6)	0.90476030	0.90467294
(0.1,0.4)	0.27958631	0.27955931	(0.5,0.8)	0.55917262	0.55911863
(0.1,0.6)	0.27958631	0.27955931	(0.7,0.2)	0.45238015	0.45233647
(0.1,0.8)	0.17279384	0.17277716	(0.7,0.4)	0.73196646	0.73189578
(0.3,0.2)	0.45238015	0.45233647	(0.7,0.6)	0.73196646	0.73189578
(0.3,0.4)	0.73196646	0.73189578	(0.7,0.8)	0.45238015	0.45233647
(0.3,0.6)	0.73196646	0.73189578	(0.9,0.2)	0.17279384	0.17277716
(0.3,0.8)	0.45238015	0.45233647	(0.9,0.4)	0.27958631	0.27955931
(0.5,0.2)	0.55917262	0.55911863	(0.9,0.6)	0.27958631	0.27955931
(0.5,0.4)	0.90476030	0.90467294	(0.9,0.8)	0.17279384	0.17277716

Table 6: Comparison of HWCM solution and exact solution of Example 2 at $t = 0.1$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.16438263	0.16435072	(0.5,0.6)	0.86071863	0.86055152
(0.1,0.4)	0.26597668	0.26592505	(0.5,0.8)	0.53195337	0.53185009
(0.1,0.6)	0.26597668	0.26592505	(0.7,0.2)	0.43035931	0.43027576
(0.1,0.8)	0.16438263	0.16435072	(0.7,0.4)	0.69633600	0.69620081
(0.3,0.2)	0.43035931	0.43027576	(0.7,0.6)	0.69633600	0.69620081
(0.3,0.4)	0.69633600	0.69620081	(0.7,0.8)	0.43035931	0.43027576
(0.3,0.6)	0.69633600	0.69620081	(0.9,0.2)	0.16438263	0.16435072
(0.3,0.8)	0.43035931	0.43027576	(0.9,0.4)	0.26597668	0.26592505
(0.5,0.2)	0.53195337	0.53185009	(0.9,0.6)	0.26597668	0.26592505
(0.5,0.4)	0.86071863	0.86055152	(0.9,0.8)	0.16438263	0.16435072

Table 7: Comparison of HWCM solution and exact solution of Example 3 at $t = 0.01$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.89302246	0.89302246	(0.5,0.6)	0.66611648	0.66611648
(0.1,0.4)	0.66976684	0.66976684	(0.5,0.8)	0.33305824	0.33305824
(0.1,0.6)	0.44651123	0.44651123	(0.7,0.2)	1.62719301	1.62719301
(0.1,0.8)	0.22325561	0.22325561	(0.7,0.4)	1.22039476	1.22039476
(0.3,0.2)	1.09074009	1.09074009	(0.7,0.6)	0.81359650	0.81359650
(0.3,0.4)	0.81805507	0.81805507	(0.7,0.8)	0.40679825	0.40679825
(0.3,0.6)	0.54537005	0.54537005	(0.9,0.2)	1.98745803	1.98745803
(0.3,0.8)	0.27268502	0.27268502	(0.9,0.4)	1.49059352	1.49059352
(0.5,0.2)	1.33223296	1.33223296	(0.9,0.6)	0.99372901	0.99372901
(0.5,0.4)	0.99917472	0.99917472	(0.9,0.8)	0.49686451	0.49686451

Table 8: Comparison of HWCM solution and exact solution of Example 3 at $t = 0.05$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.92946740	0.92946739	(0.5,0.6)	0.69330121	0.69330121
(0.1,0.4)	0.69710055	0.69710055	(0.5,0.8)	0.34665060	0.34665060
(0.1,0.6)	0.46473370	0.46473370	(0.7,0.2)	1.69360001	1.69360001
(0.1,0.8)	0.23236685	0.23236685	(0.7,0.4)	1.27020001	1.27020001
(0.3,0.2)	1.13525404	1.13525404	(0.7,0.6)	0.84680001	0.84680001
(0.3,0.4)	0.85144053	0.85144053	(0.7,0.8)	0.42340000	0.42340000
(0.3,0.6)	0.56762702	0.56762702	(0.9,0.2)	2.06856773	2.06856773
(0.3,0.8)	0.28381351	0.28381351	(0.9,0.4)	1.55142580	1.55142580
(0.5,0.2)	1.38660241	1.38660241	(0.9,0.6)	1.03428386	1.03428386
(0.5,0.4)	1.03995181	1.03995181	(0.9,0.8)	0.51714193	0.51714193

Table 9: Comparison of HWCM solution and exact solution of Example 3 at $t = 0.1$

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.97712221	0.97712221	(0.5,0.6)	0.72884752	0.72884752
(0.1,0.4)	0.73284166	0.73284165	(0.5,0.8)	0.36442376	0.36442376
(0.1,0.6)	0.48856110	0.48856110	(0.7,0.2)	1.78043274	1.78043274
(0.1,0.8)	0.24428055	0.24428055	(0.7,0.4)	1.33532456	1.33532456
(0.3,0.2)	1.19345976	1.19345976	(0.7,0.6)	0.89021637	0.89021637
(0.3,0.4)	0.89509482	0.89509482	(0.7,0.8)	0.44510819	0.44510819
(0.3,0.6)	0.59672988	0.59672988	(0.9,0.2)	2.17462546	2.17462546
(0.3,0.8)	0.29836494	0.29836494	(0.9,0.4)	1.63096910	1.63096910
(0.5,0.2)	1.45769504	1.45769504	(0.9,0.6)	1.08731273	1.08731273
(0.5,0.4)	1.09327128	1.09327128	(0.9,0.8)	0.54365637	0.54365637

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