International Research Journal of Mathematics, Engineering and IT
Vol. 3, Issue 7, July 2016 IF- 3.563 ISSN: (2349-0322)

Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

# TWO STAGE LIU REGRESSION ESTIMATOR

**Issam Dawoud[1] and Selahattin Kaçıranlar[1]**
[1]Department of Statistics, Faculty Science and Letters, Çukurova University, Adana, Turkey.

## ABSTRACT

*This paper introduces a new estimator for multicollinearity and autocorrelated errors. We propose the Two Stages Liu estimator (TL) for the multiple linear regression model which suffers from autocorrelation AR(1) and multicollinearity problems. We use a mixed method to apply the two stages least squares procedure (TS) for deriving the TL estimator. Furthermore, a Monte Carlo study and a real data are carried out to investigate the performance of the proposed estimator over the others.*

**Keywords**: Two Stages Estimator; Multicollinearity; Matrix Mean Square Error; Autocorrelated Errors; General Linear Models

## 1. Introduction

Consider a multiple linear regression model of the form

$$Y = X\beta + \varepsilon,\ \varepsilon \sim (0, \sigma^2 \mathrm{I}_n) \tag{1}$$

where $Y$ is an $n \times 1$ vector of observations on the dependent variable, $X$ is an $n \times p$ known design matrix of rank $p$, $\beta$ is an $p \times 1$ vector of unknown parameters, and $\varepsilon$ is an $n \times 1$ vector of random errors with zero mean and variance $\sigma^2 \mathrm{I}_n$ , where $\mathrm{I}_n$ is an identity matrix of order $n$ .

The ordinary least squares (OLS) estimator of $\beta$ is defined as:

$$\hat{\beta} = (X'X)^{-1} X'Y \,. \tag{2}$$

Both the OLS estimator and its covariance matrix heavily depend on the characteristics of the $X'X$ matrix. If $X'X$ is ill-conditioned, i.e. the column vectors of $X$ are linearly dependent, the OLS estimators are sensitive to a number of errors. For example, some of the regression coefficients may be statistically insignificant or have the wrong sign, and they may result in wide confidence intervals for individual parameters. With ill conditioned $X'X$ matrix, it is difficult to make valid statistical inferences about the regression parameters. One of the most popular estimator dealing with multicollinearity is the ordinary ridge regression (ORR) estimator proposed by Hoerl and Kennard (1970a) and is defined as

$$\hat{\beta}_k = (X'X + k\mathrm{I}_p)^{-1} X'Y = (\mathrm{I}_p + k(X'X)^{-1})^{-1} \hat{\beta}\,, \tag{3}$$

where the constant $k > 0$ is known as the biasing parameter.

Another biased estimator is Stein (SLS) estimator which is given as (see, Stein (1956), James and Stein (1961)):

$$\hat{\beta}_s = c\,\hat{\beta}\,, \tag{4}$$

which it is a linear function of $c$ and $0 < c < 1$, where $c = (1 - \frac{a}{n}\frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{\hat{\beta}'X'X\hat{\beta}})$ and $a \geq 0$ is the shrinkage factor.

The Liu estimator (LE) is defined, see for example Liu (1993), Akdeniz and Kaçıranlar (1995) and Kaçıranlar et. al. (1999), as follows

$$\hat{\beta}_d = (X'X + \mathrm{I}_p)^{-1} (X'Y + d\hat{\beta}) = (X'X + \mathrm{I}_p)^{-1} (X'X + d\mathrm{I}_p)\hat{\beta} \tag{5}$$

where $d$ is a constant, such that $0 < d < 1$.

The advantage of the LE estimator over the ORR estimator is that the LE estimator is a linear function of *d*, so it is easy to choose *d* than to choose *k* in the ORR estimator.

Since the matrix $X'X$ is symmetric, it exists an orthogonal matrix $U=[U_1,U_2,...,U_p]$, such that $U'(X'X)U = diag(\lambda_1,\lambda_2,...,\lambda_p)=\Lambda$, where the $\lambda_i$ is the i$^{th}$ eigenvalue of $X'X$, and the columns of $U$ are normalized eigenvectors associated with eigenvalues. Thus, the model $Y = X\beta+\varepsilon$ can be written in the canonical form as:

$$Y = Z\alpha+\varepsilon\,, \tag{6}$$

where $Z = XU$ and $\alpha = U'\beta$. The OLS, ORR, SLS and LE estimators for (6) are respectively given as:

$$\begin{aligned} \hat{\alpha} &= \Lambda^{-1}Z'Y \\ \hat{\alpha}_k &= (\mathrm{I}_p + k\,\Lambda^{-1})\ \hat{\alpha} \\ \hat{\alpha}_S &= c\,\hat{\alpha} \\ \hat{\alpha}_d &= (\Lambda+\mathrm{I}_p)^{-1}(\Lambda+d\,\mathrm{I}_p)\hat{\alpha} \end{aligned} \quad . \tag{7}$$

Let us consider the multiple linear regression model

$$Y = X\beta+\varepsilon,\ \varepsilon \sim (0,\sigma^2 V)\,. \tag{8}$$

Aitken (1935) derived the generalized least squares (GLS) estimator as:

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}Y\,, \tag{9}$$

where $V$ is a known positive definite (p.d.) matrix .

Trenkler (1984) proposed the ridge estimator of $\beta$ in the general linear regression model (GRR) as:

$$\hat{\beta}_{GRR} = (X'V^{-1}X + k\,\mathrm{I}_p)^{-1} X'V^{-1}Y\,. \qquad (10)$$

He concluded that the Ridge Regression estimators which take the autocorrelation into account can perform better than the other methods when $V$ matrix is known.

Stein (1975) proposed the Generalized Stein (GS) estimator of $\beta$ in the general linear regression model as:

$$\hat{\beta}_{GS} = f\hat{\beta}_{GLS}\,, \qquad (11)$$

which it is a linear function of $0 < f < 1$, where $f = (1 - \dfrac{a}{n}\dfrac{(Y - X\hat{\beta}_{GLS})'V^{-1}(Y - X\hat{\beta}_{GLS})}{\hat{\beta}_{GLS}X'V^{-1}X\hat{\beta}_{GLS}})$.

Kaçıranlar (2003) combined the Liu estimator of Equation (5) with the GLS of Equation (9) to obtain the Generalized Liu estimator (GLE) which is defined as:

$$\hat{\beta}_{GLE} = (X'V^{-1}X + \mathrm{I}_p)^{-1}(X'V^{-1}X + d\,\mathrm{I}_p)\hat{\beta}_{GLS}\,. \qquad (12)$$

So, the problem of multicollinearity has also been discussed when the violation of the assumption of the autocorrelation of errors is also faced by many researchers, see for example, Gosling et al. (1982), Firinguetti (1989), Bayhan and Bayhan (1998), Kaçıranlar (2003), Özkale (2008), Alheety and Kibria (2009), Güler and Kaçıranlar (2009), Şiray et al. (2014) and Özkale (2014).

Using the canonical form, The GLS, GRR, GS and GLE are respectively given as:

$$\begin{aligned}
\hat{\alpha}_{GLS} &= \Gamma^{-1}Q'X'V^{-1}Y \\
\hat{\alpha}_{GRR} &= (\mathrm{I}_p + k\,\Gamma^{-1})\,\hat{\alpha}_{GLS} \\
\hat{\alpha}_{GS} &= f\,\hat{\alpha}_{GLS} \\
\hat{\alpha}_{GLE} &= (\Gamma + \mathrm{I}_p)^{-1}(\Gamma + d\,\mathrm{I}_p)\,\hat{\alpha}_{GLS}
\end{aligned}\,, \qquad (13)$$

where $Q'(X'V^{-1}X)Q = diag(\gamma_1, \gamma_2, ..., \gamma_p) = \Gamma$, $\gamma_i$ is the i$^{th}$ eigenvalue of $X'V^{-1}X$ .

In this paper, we introduce a new estimator which is called the Two Stages Liu estimator (TL) by mixing the Two Stages procedure (TS) with the LE estimator. So, we examine the multicollinearity and autocorrelation problems simultaneously, define the TL estimator in the linear regression model with $AR(1)$ correlated errors, and find the characteristics of this estimator in Sect. 2. Then, in Sect. 3, some interesting transforms of the TL estimator will be discussed. Then, we give a simulation study in Sect. 4. Finally, we give an application of a real data in Sect. 5.

## 2. The Two Stages Liu Estimator (TL)

The model with first order autoregressive process $AR(1)$ has the form:

$$\varepsilon_t = \rho \varepsilon_{t-1} + \eta_t, \; t = 2,3,...,n \tag{14}$$

where $\rho$ is the autocorrelation parameter (coefficient) ($|\rho| < 1$), $\eta_t$ is a normal distributed random variable, which satisfies

$$\eta_t \sim N(0, \sigma^2), \quad E(\eta_t \eta_{t-s}) = \begin{cases} \sigma^2, & if\ s = 0 \\ 0, & else. \end{cases} \tag{15}$$

If $V$ is an $n \times n$ known p.d. symmetric matrix, the simplest solution to the estimated model (8) when plagued with the problems of multicollinearity and autocorrelation in errors, is the use of GLS as in (9), but $V$ matrix is seldom known. If $V$ matrix is unknown, it is common in practice to use the estimated matrix of $V$ in order to find the estimated generalized least square estimator (EGLSE) or Two Stages method estimator that is more efficient than the GLSE.

So, let us reform the Two Stages procedure.

Using the matrix $\mathbf{P}$ to transform the model in (8) yields

$$\mathbf{P}Y = \mathbf{P}X\beta + \mathbf{P}\varepsilon \,,$$

which is equivalent to

$$Y^* = X^*\beta + \varepsilon^* \,, \tag{16}$$

where $E(\varepsilon^*) = 0$ and $Cov(\varepsilon^*) = \sigma^2 \mathrm{I}_n$. Therefore, the OLS estimator for the model (16) is:

$$\hat{\beta}_{TS} = (X^{*\prime}X^*)^{-1}X^{*\prime}Y^* \tag{17}$$

where

$$Y^* := \mathbf{P}Y = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ \vdots \\ Y_n \end{pmatrix}$$

$$X^* := \mathbf{P}X = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1p-1} \\ 1 & X_{21} & X_{22} & \cdots & X_{2p-1} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{np-1} \end{pmatrix}$$

Note that $X^{*\prime}X^* = X'\mathbf{P}'\mathbf{P}X = X'V^{-1}X$ and $X^{*\prime}Y^* = X'\mathbf{P}'\mathbf{P}Y = X'V^{-1}Y$**,** where

$$V^{-1} := \mathbf{P}'\mathbf{P} = \begin{pmatrix} 1 & -\rho & 0 & \cdots & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & \ddots & \ddots & \vdots \\ 0 & -\rho & 1+\rho^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1+\rho^2 & -\rho \\ 0 & \cdots & \cdots & 0 & -\rho & 1 \end{pmatrix} \tag{18}$$

and $\hat{V}^{-1}$ is the estimated matrix that is in (18) with $\rho$ replaced by $\hat{\rho}$.

A number of $\hat{\rho}$ 's alternatives have been used in the literature, Judge, et. al. (1985):

1. The sample correlation coefficient. In this case, the estimator is

$$r_1 = \frac{\sum_{t=2}^{n} \hat{\varepsilon}_t \hat{\varepsilon}_{t-1}}{\sum_{t=1}^{n} \hat{\varepsilon}_t^2} , \tag{19}$$

where the disturbances ($\varepsilon_t$'$s$), because they are unobservable, have been replaced by the OLS residuals $\hat{\varepsilon}_t = Y_t - \hat{Y}_t = Y_t - x_t'\hat{\beta}$, $t = 1, 2, ..., n$, where $x_t'$ is the symbol of a $(1 \times p)$ vector containing the *t*-th observation on $p$ predictors.

2. The Durbin-Watson Statistic. This statistic,

$$d = \frac{\sum_{t=2}^{n} (\hat{\varepsilon}_t - \hat{\varepsilon}_{t-1})^2}{\sum_{t=1}^{n} \hat{\varepsilon}_t^2} . \tag{20}$$

is often used to test the existence of autocorrelation.

An estimator for $\rho$ that is approximately equal to $r_1$, namely

$$\hat{\rho} = 1 - 0.5\,d\,. \tag{21}$$

So, after estimated $\rho$ by $\hat{\rho}$, we can find $\hat{V}^{-1}$. And then, the TS is given as (Prais and Winsten (1954)):

$$\hat{\beta}_{TS} = (X'\hat{V}^{-1}X)^{-1}X'\hat{V}^{-1}Y\,. \tag{22}$$

Eledum and Zahri (2013) proposed the TR estimator of $\beta$ in the general linear regression model as:

$$\hat{\beta}_{TR} = (X'\hat{V}^{-1}X + k\,\mathrm{I}_p)^{-1}X'\hat{V}^{-1}Y\,. \tag{23}$$

Chaturvedi and Shukla (1990) proposed the Two Stages Stein (STS) estimator of $\beta$ in the general linear regression model as:

$$\hat{\beta}_{STS} = \hat{f}\hat{\beta}_{TS}\,, \tag{24}$$

which it is a linear function of $\hat{f}$ and $0 < \hat{f} < 1$, where $\hat{f} = (1 - \frac{a}{n}\frac{(Y - X\hat{\beta}_{TS})'\hat{V}^{-1}(Y - X\hat{\beta}_{TS})}{\hat{\beta}_{TS}X'\hat{V}^{-1}X\hat{\beta}_{TS}})$.

To estimate the linear model with both multicollinearity and autocorrelation *AR(1)* problems simultaneously, we propose the mixed estimator, which is developed by mixing Equation (5) with (22). Therefore, the TL estimator is

$$\begin{aligned}\hat{\beta}_{TL} &= (X^{*\prime}X^{*} + \mathrm{I}_p)^{-1}(X^{*\prime}X^{*} + d\,\mathrm{I}_p)\,\hat{\beta}_{TS} \\ &= (X'\mathbf{P}'\mathbf{P}X + \mathrm{I}_p)^{-1}(X'\mathbf{P}'\mathbf{P}X + d\,\mathrm{I}_p)\,\hat{\beta}_{TS} \\ &= (X'\hat{V}^{-1}X + \mathrm{I}_p)^{-1}(X'\hat{V}^{-1}X + d\,\mathrm{I}_p)\,\hat{\beta}_{TS}\end{aligned} \tag{25}$$

where $0 < d < 1$ and $\hat{V}^{-1}$ is the estimated $V^{-1}$ matrix which it is defined in (18).

In order to compare the performance of any estimator with others, a criterion for measuring the goodness of an estimator is required. For this purpose, the matrix mean square error (MMSE) criterion is used to measure the goodness of an estimator. We note that for any estimator $\tilde{\beta}$ of $\beta$, its MMSE is defined as

$$MSE(\tilde{\beta}) = E(\tilde{\beta} - \beta)(\tilde{\beta} - \beta)' = Cov(\tilde{\beta}) + Bias(\tilde{\beta})Bias(\tilde{\beta})' \qquad (26)$$

and the scalar mean square error (mse) is obtained as follows

$$mse(\tilde{\beta}) = \mathrm{tr}(\mathrm{MSE}(\tilde{\beta})). \qquad (27)$$

**3. Some Interesting Transforms of the TL Estimator**

In this section, we use some properties of the symmetrical matrices to improve the results above by using the eigenvalues and the eigenvectors. Recall that $X'\hat{V}^{-1}X$ is a symmetric matrix (correlation form), therefore it exists an orthogonal matrix $Q$ such that

$$Q'(X'\hat{V}^{-1}X)Q = diag(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_p) = \hat{\Gamma}, \qquad (28)$$

where $\hat{\gamma}_i$ is the i$^{\text{th}}$ eigenvalue of the matrix $X'\hat{V}^{-1}X$, the columns of $Q$ are normalized eigenvectors associated with the eigenvalues. Thus, Eledum and Zahri (2013) rewrote the TS estimator and the TR estimator respectively as follows:

$$\hat{\beta}_{TS} = Q\hat{\Gamma}^{-1}Q' r_{X^*Y^*} = \sum_{i=1}^{p} \hat{\gamma}_i^{-1} Q_j Q_j^{'} r_{X^*Y^*}, \qquad (29)$$

$$\hat{\beta}_{TR} = Q(\hat{\Gamma} + k\mathrm{I}_p)^{-1} Q' r_{X^*Y^*} = \sum_{i=1}^{p} (\hat{\gamma}_i + k)^{-1} Q_j Q_j^{'} r_{X^*Y^*}, \qquad (30)$$

where $r_{X^*Y^*}$ is the correlation matrix between $X^*$ and $Y^*$ and $Q_j$ represents the j[th] column of the orthogonal matrix $Q$.

Thus, we can rewrite the STS estimator and the proposed TL estimator respectively as follows:

$$\hat{\beta}_{STS} = \hat{f}\, Q\hat{\Gamma}^{-1}Q' r_{X^*Y^*} = \hat{f}\sum_{i=1}^{p}\hat{\gamma}_i^{-1}Q_j Q_j^{'} r_{X^*Y^*}, \tag{31}$$

$$\hat{\beta}_{TL} = Q(\hat{\Gamma}+\mathrm{I}_p)^{-1}(\hat{\Gamma}+d\mathrm{I}_p)\hat{\Gamma}^{-1}Q' r_{X^*Y^*} = \sum_{i=1}^{p}(\hat{\gamma}_i+1)^{-1}(\hat{\gamma}_i+d)\,\hat{\gamma}_i^{-1}\,Q_j Q_j^{'} r_{X^*Y^*}. \tag{32}$$

Using canonical form:

$$Y^* = Z\,\alpha^* + \varepsilon^*, \tag{33}$$

where $Z = X^*Q$ and $\alpha^* = Q'\beta$. Thus, Eledum and Zahri (2013) rewrote the TS estimator and the TR estimator respectively as follows:

$$\hat{\alpha}_{TS}^{*} = (Z'Z)^{-1}Z'Y^* = \hat{\Gamma}^{-1}Q'X'\hat{V}^{-1}Y, \tag{34}$$

$$\hat{\alpha}_{TR}^{*} = (\hat{\Gamma}+k\mathrm{I}_p)^{-1}X'\hat{V}^{-1}Y = C_1Y. \tag{35}$$

The STS estimator rewrite as follows:

$$\hat{\alpha}_{STS}^{*} = \hat{f}\,\hat{\alpha}_{TS}^{*}. \tag{36}$$

Thus, we can rewrite the proposed TL estimator as follows:

$$\hat{\alpha}_{TL}^{*} = (\hat{\Gamma}+\mathrm{I}_p)^{-1}(\hat{\Gamma}+d\mathrm{I}_p)\hat{\alpha}_{TS}^{*}, \tag{37}$$

$$\hat{\alpha}_{TL}^{*} = (\hat{\Gamma}+\mathrm{I}_p)^{-1}(\hat{\Gamma}+d\mathrm{I}_p)\hat{\Gamma}^{-1}X'\hat{V}^{-1}Y = C_2Y. \tag{38}$$

For practical purposes, we have to replace these unknown parameters by some suitable estimates. Liu (1993) gave the estimates of $d$ by analogy with the estimate of $k$ in ridge estimators. Following the method of Liu (1993), some of these estimates are defined as

$$\hat{d}_{mm} = 1 - \hat{\sigma}_{TS}^2 \left[ \sum_{i=1}^{p} \frac{1}{\hat{\gamma}_i (\hat{\gamma}_i + 1)} \Big/ \sum_{i=1}^{p} \frac{\hat{\alpha}_{TSi}^{*2}}{(\hat{\gamma}_i + 1)^2} \right], \tag{39}$$

$$\hat{d}_{mmh} = 1 - h\hat{\sigma}_{TS}^2 \left[ \sum_{i=1}^{p} \frac{1}{\hat{\gamma}_i (\hat{\gamma}_i + 1)} \Big/ \sum_{i=1}^{p} \frac{\hat{\alpha}_{TSi}^{*2}}{(\hat{\gamma}_i + 1)^2} \right], \tag{40}$$

$$\hat{d}_{CL} = 1 - \hat{\sigma}_{TS}^2 \left[ \sum_{i=1}^{p} \frac{1}{(\hat{\gamma}_i + 1)} \Big/ \sum_{i=1}^{p} \frac{\hat{\gamma}_i \, \hat{\alpha}_{TSi}^{*2}}{(\hat{\gamma}_i + 1)^2} \right], \tag{41}$$

$$\hat{d}_{CLh} = 1 - h\hat{\sigma}_{TS}^2 \left[ \sum_{i=1}^{p} \frac{1}{(\hat{\gamma}_i + 1)} \Big/ \sum_{i=1}^{p} \frac{\hat{\gamma}_i \, \hat{\alpha}_{TSi}^{*2}}{(\hat{\gamma}_i + 1)^2} \right], \tag{42}$$

where $h > 0$ and $\hat{\alpha}_{TS}^*$ and $\sigma_{TS}^2$ are the TS estimates of $\alpha$ and $\sigma^2$.

A very important issue in the study of ridge regression is how to find an appropriate parameter $k$. When $k$ is estimated from the data the ridge estimator is called the operational ridge estimator. Hoerl and Kennard (1970 a, b), Hoerl, Kennard and Baldwin (1975) and Lawless and Wang (1976) suggested some of the operational ridge parameters. Following them, some of these estimates are defined as

$$\hat{k}_{HK} = \frac{\hat{\sigma}_{TS}^2}{\sum_{i=1}^{p} \hat{\alpha}_{TSi}^{*2}}, \tag{43}$$

$$\hat{k}_{HKB} = \frac{p\,\hat{\sigma}_{TS}^2}{\sum_{i=1}^{p} \hat{\alpha}_{TSi}^{*2}}, \tag{44}$$

$$\hat{k}_{LW} = \frac{p\,\hat{\sigma}_{TS}^2}{\sum_{i=1}^{p} \hat{\gamma}_i \, \hat{\alpha}_{TSi}^{*2}}. \tag{45}$$

## 4. The Monte Carlo Simulation Study

In this section, we will discuss the simulation study to compare the performances of the *OLS*, TS, ORR, TR, SLS, STS, Liu and TL estimators. MATLAB is used for the simulation experiment. Following McDonald and Galarneau (1975) and Kibria (2003), the explanatory variables are generated by

$$x_{ij} = (1-\gamma^2)^{1/2} z_{ij} + \gamma z_{i\,p+1} \;,\; i = 1,2,..., n, \quad j = 1,2,..., p \tag{46}$$

where $z_{ij}$ are independent standard normal pseudo-random numbers, $\gamma$ is specified so that the correlation between any two explanatory variables is given by $\gamma^2$. Following Kibria (2003), three different sets of correlation are considered, corresponding to $\gamma = 0.7, 0.8, 0.9$. The explanatory variables are then standardized so that $X'X$ is in the correlation form. The $\hat{V}^{-1}$ matrix is the estimated $V^{-1}$ matrix which is defined in (18). Using the second method of $\rho$'s estimation in $AR(1)$, five values of Durbin-Watson statistic are taken as $d = 0.2, 0.6, 1.0, 1.4, 1.8$. So, the estimated five different values of $\rho$ are $\hat{\rho} = 0.1, 0.3, 0.5, 0.7, 0.9$. Following Şiray et al. (2014), we choose the $\beta$ as the eigenvector corresponding to the largest and the smallest eigenvalue of the matrix $X'\hat{V}^{-1}X$. Observations on the dependent variable are determined by

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \varepsilon_i, \quad i = 1,2,\ldots,n \tag{47}$$

where $\varepsilon_i$ are independent normal pseudo-random numbers with mean 0 and variance $\sigma^2\hat{V}$ and $\beta_0$ is taken to be identically zero. Six values of $\sigma$ are considered which are 0.1, 0.5, 1, 4, 9 and 20. Then the dependent variable is standardized so that $X'y$ is the vector of correlations of the dependent variable with each explanatory variable. Where the biasing parameter $k$ in ORR and TR estimators and the biasing parameter $d$ in Liu and TL estimators are chosen as 0.001, 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9 and 1. Moreover, the biasing parameters $k$ and $d$ are taking as given in equations 47-53. Also, the constant $a$ in SLS and STS estimators is chosen as $a = p-3$ (see

Chaturvedi, et. al., 2001). In this study, we choose $n$ =20 and 60 and $p$ =4. Then the experiment is replicated 5,000 times by generating new error terms.

We use the SMSE criterion to investigate the performance of the OLS, TS, ORR, TR, SLS, STS, Liu and TL estimators. The estimated SMSE for any estimator $\hat{\beta}*$ is calculated as follows:

$$m\hat{s}e = \frac{1}{MCN}\sum_{r=1}^{MCN}(\hat{\beta}_r * - \beta)'(\hat{\beta}_r * - \beta) \qquad (48)$$

where $\hat{\beta}_r *$ is the computed value of $\hat{\beta}*$ for the $r^{\text{th}}$ replication of the experiment and $MCN$ is the number of replications, which is taken 5000 for this experiment.

The results of the simulation study are summarized in Tables 1–6. We have the following comments. Note that for each case we chose the best $k$ in ORR and TR estimators and the best $d$ in Liu and TL estimators among all suggested $k$'s and $d$'s which give the smallest SMSE. Firstly, we comment about $\sigma$. As $\sigma$ increases, the estimated SMSEs of the mentioned estimators also increase as expected (e.g. for $\hat{\rho} = 0.1, \gamma = 0.8, n = 20, 60$, the SMSEs of the suggested estimators at $\sigma = 1$ are larger than the SMSEs of the suggested estimators at $\sigma = 0.1$). As $\gamma$ increases, the estimated SMSEs of the mentioned estimators also increase as expected (e.g. for $\hat{\rho} = 0.1, n = 20, 60, \sigma = 1$, the SMSEs of the suggested estimators at $\gamma = 0.8$ are larger than the SMSEs of the suggested estimators at $\gamma = 0.7$ ). As $n$ increases, the estimated SMSEs of the mentioned estimators decrease as expected (e.g. for $\hat{\rho} = 0.1, \gamma = 0.7, \sigma = 1$, the SMSEs of the suggested estimators at $n = 60$ are smaller than the SMSEs of the suggested estimators at $n = 20$). Now, we investigate the effect of $\gamma$, which designates the degree of multicollinearity. As multicollinearity becomes more serious, inflation in SMSE of the OLS and TS estimators is expected. An increase in $\gamma$, is an increase in the estimated SMSE of the OLS and TS estimators, expectedly. For $\gamma$ =*0.7,0.8,0.9,* while $\hat{\rho}$ is increasing, the estimated SMSEs of the OLS, ORR, Stein and Liu estimators are increasing and the estimated SMSEs of the TS, TR, STS and TL estimators are decreasing. Also, when $\gamma$ is increasing, the best model is the Liu estimator which its SMSE is decreasing rapidly rather than ORR estimator but when $\gamma$ and $\hat{\rho}$ are increasing simultaneously, the best model is the TL estimator which its SMSE is decreasing rapidly rather than the TR estimator. So, we can say that the STS estimator gives closed or better results than the TL estimator according to SMSE values for larger values of $\sigma$.

## 5. The Real Life Data Study

To illustrate the performance of the estimators, we consider the famous Portland cement data originally due to Woods et al. (1932). This data have been analyzed by several researchers: Hald (1952, pp. 635–652), Hamaker (1962), Gorman and Toman (1966, pp. 35–36), Daniel and Wood (1980, pp. 89–91, 106 107), Nomura (1988, pp. 735), Piepel and Redgate (1998) and Kaçıranlar et. al. (1999), Liu (2003), Sakallıoglu and Kaçıranlar (2008), and very recently, Muniz and Kibria (2009), among others. The data came from an experimental investigation of the heat evolved during the setting and hardening of Portland cement of varied composition and the dependence of this heat on the percentages of four compounds in the clinkers from which the cement was made. There are four explanatory variables: $X_1$: amount of tricalcium aluminate, $X_2$: amount of ticalcium silicate, $X_3$: amount of tetracalcium alumino ferrite, and $X_4$: amount of dicalcium silicate. The response variable is $Y$: heat evolved in calories per gram of cement.

Consider the following linear regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon , \tag{49}$$

where $Y$ represents the dependent variable and $X_i$ for i= 1, 2, 3 and 4 are the independent variables.

The fitted model is:

$$\hat{Y} = 62.4054 + 1.5511 X_1 + 0.5102 X_2 + 0.1019 X_3 - 0.1441 X_4 . \tag{50}$$

**Table 7**. Output using original data

| $\alpha$ | $n$ | $p$ | $\hat{\sigma}^2$ | $dl$ | $du$ | $DW$ | $VIF_1$ | $VIF_2$ | $VIF_3$ | $VIF_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 13 | 4 | 5.983 | 0.574 | 2.094 | 2.053 | 38.496 | 254.423 | 46.868 | 282.513 |

where $VIF_i$ for each i= 1, 2, 3 represents the Variance Inflation Factors.

Table 7 shows that $dl < DW < du$, that means, the test is inconclusive. So, we took the residuals of the estimated model and we created the model of the residuals by its lags such that the best model of residuals is $AR(1)$ model with $\hat{\rho} = -0.081$ or by using eq.(19). So, the model suffers from first order autoregressive scheme and since all $VIFs > 4$, the model also suffers from multicollinearity.

The $X'X$ (correlation form) is:

$$r_{X'X} = \begin{pmatrix} 1 & 0.229 & -0.824 & -0.245 \\ 0.229 & 1 & -0.139 & -0.973 \\ -0.824 & -0.139 & 1 & 0.030 \\ -0.245 & -0.973 & 0.030 & 1 \end{pmatrix}$$

**Table 8**. Output using transformed data

| $VIF_1$ | $VIF_2$ | $VIF_3$ | $VIF_4$ |
|---|---|---|---|
| 11.322 | 102.586 | 12.575 | 116.204 |

So, we solved the autocorrelated error using TS estimator, where the new $\hat{\sigma}^2 = 5.923$ of a transformed data. Table 8 shows that all $VIFs > 4$, that means, the model still suffers from the multicollinearity problem.

The corresponding $X'\hat{V}^{-1}X$ (correlation form) is:

$$r_{X'\hat{V}^{-1}X} = \begin{pmatrix} 1 & 0.245 & -0.788 & -0.269 \\ 0.245 & 1 & -0.071 & -0.980 \\ -0.788 & -0.071 & 1 & -0.012 \\ -0.269 & -0.980 & -0.012 & 1 \end{pmatrix}$$

Since the model still suffers from the multicollinearity problem. We will use the TR estimator and the proposed TL estimator as the alternatives to solve this problem

Table 9 summarizes the interesting comparisons for the estimators subject to this study. See Table 10 and 11 in Appendices for the complete comparisons.

**Table 9**. Comparison of Estimators

| **Estimator** | *Var* | $Bias^2$ | *mse* | $k$ **or** $d$ |
|---|---|---|---|---|
| $\hat{\beta}_{OLS}$ | 4912.100 | 0.000000 | 4912.100000 | ------- |
| $\hat{\beta}_{TS}$ | 4647.422 | 0.000000 | 4647.422000 | ------- |
| $\hat{\beta}_{k}$ | 57.97370 | 3096.800 | 3154.774000 | $k=0.01$ |
| $\hat{\beta}_{TR}$ | 59.44948 | 2321.100 | 2380.549000 | $k=0.01$ |
| $\hat{\beta}_{S}$ | 4911.800 | 0.000004 | 4911.800004 | -------- |
| $\hat{\beta}_{STS}$ | 4649.600 | 0.000002 | 4649.600002 | -------- |
| $\hat{\beta}_{d}$ | 788.8650 | 1399.700 | 2188.565000 | $d=0.4$ |
| $\hat{\beta}_{TL}$ | 746.4901 | 1059.600 | 1806.090000 | $d=0.4$ |

Table 9 shows that $\hat{\beta}_{TS}$ is better than $\hat{\beta}_{OLS}$ because $\hat{\beta}_{TS}$ gives smaller mse value than $\hat{\beta}_{OLS}$ which means, the correlation among errors is occurred. Also, the other biased estimators in general give better results in terms of the mse values than the $\hat{\beta}_{OLS}$ and $\hat{\beta}_{TS}$ estimators which means, the data also suffers from multicollinearity problem as we mentioned above. Moreover, it shows clearly the good results of our improved estimator $\hat{\beta}_{TL}$ which gives the smallest mse value among all the mentioned estimators especially when $d=0.4$.

According to the Tables 10 and 11 in Appendices, $\hat{\beta}_{TL}$ is better than the other estimators in general and it is better than $\hat{\beta}_{TR}$ when $d = 0.2, 0.3, 0.4, 0.5, 0.6$ in terms of the mse criterion.

**Table 12.** Comparisons between the ORR and the TR estimators with different estimated biasing parameter $k$

| **Estimator** | **The estimated biasing parameter** $k$ | $Var$ | $Bias^2$ | $mse$ |
|---|---|---|---|---|
| $\hat{\beta}_k$ | $\hat{k}_{HK} = 0.0015$ | 961.4145 | 1211.70000 | 2173.11450000 |
| | $\hat{k}_{HKB} = 0.0077$ | 92.18690 | 2902.80000 | 2994.98690000 |
| | $\hat{k}_{LW} = 1.4889 \times 10^{-7}$ | 4910.90 | 0.00005821 | 4910.90005821 |
| $\hat{\beta}_{TR}$ | $\hat{k}_{HK} = 0.0020$ | 703.5970 | 1105.10000 | 1808.69700000 |
| | $\hat{k}_{HKB} = 0.01$ | 59.37540 | 2329.20000 | 2388.57540000 |
| | $\hat{k}_{LW} = 1.7136 \times 10^{-7}$ | 4648.600 | 0.00005355 | 4648.60005355 |

Table 12 shows that the mse values for the ORR and the TR estimator such that the mse values of the TR estimator are always smaller than the mse values of the ORR estimator for the three estimated biasing parameter $k$. (i.e. for $\hat{k}_{HK}$, the mse value of the TR estimator is smaller than the mse value of the ORR estimator and etc.)

**Table 13.** Comparisons between the Liu and the TL estimators with different estimated biasing parameter $d$

| **Estimator** | **The estimated biasing parameter** $d$ | $Var$ | $Bias^2$ | $mse$ |
|---|---|---|---|---|
| $\hat{\beta}_d$ | $\hat{d}_{mm} = 0.9939$ | 4852.100 | 0.14660 | 4852.2466 |

| | | | | |
|---|---|---|---|---|
| | $\hat{d}_{mm(h=4.5)} = 0.9724$ | 4644.700 | 2.96870 | 4647.6687 |
| | $\hat{d}_{CL} = 0.8702$ | 3721.400 | 65.4625 | 3786.8625 |
| | $\hat{d}_{CL(h=0.45)} = 0.4161$ | 853.4019 | 1325.60 | 2179.0019 |
| $\hat{\beta}_{TL}$ | $\hat{d}_{mm} = 0.9920$ | 4575.800 | 0.18920 | 4575.9892 |
| | $\hat{d}_{mm(h=4.5)} = 0.9640$ | 4321.300 | 3.83120 | 4325.1312 |
| | $\hat{d}_{CL} = 0.8303$ | 3207.300 | 85.0348 | 3292.3348 |
| | $\hat{d}_{CL(h=4.5)} = 0.2364$ | 261.9467 | 1722.00 | 1983.000 |

Table 13 shows that the mse values for the Liu and the TL estimator such that the mse values of the TL estimator are always smaller than the mse values of the Liu estimator for the four estimated biasing parameter $d$. (i.e. for $\hat{d}_{CL}$, the mse value of the TL estimator is smaller than the mse value of the Liu estimator and etc.).

Finally, according to the Tables 9, 12, and 13, we see that the TL estimator is the best estimator which gives the smallest mse value when $d = 0.4$ comparing to the other mentioned estimators and then the TR estimator when $\hat{k}_{HK} = 0.0020$ and so on.

## Conclusions:

In this paper, we have examined the multicollinearity and autocorrelation problem simultaneously and defined the TL estimator in the linear regression model with $AR(1)$ correlated errors. Also, the results of our simulation and real life dataset suggest us that the mse of the TL estimator is smaller than the mentioned estimators but when $\sigma$ gets larger, the STS estimator gives closed or better results than the TL estimator according to mse values.

## References


Aitken, A. C. (1935). On least squares and linear combinations of observations. Proc. R. Soc. Edinb. 55, 42–48.

Akdeniz F., Kaciranlar S., (1995). On the almost unbiased generalized Liu estimator and unbiased estimation of the bias and MSE. Communications in Statistics – Theory and Methods, 24, 1789-1797.

Alheety, M. and Kibria, B. M. (2009). On The Liu And Almost Unbiased Liu Estimators in The Presence Of Multicollinearity With Heteroscedastic Or Correlated Errors. Surveys in Mathematics and its Applications, 4, 155-167.

Bayhan, G. M. and Bayhan, M. (1998). Forecasting using autocorrelated errors and multicollinear predictor variables. Comp. Ind. Eng. 34, 2, 413–421.

Chaturvedi, A. and Shukla, G. (1990). Stein rule estimation in linear model with nonscalar error covariance matrix. Sankhya, 52, B, 293-304.

Chaturvedi, A., Wan, A. and Singh, Sh. (2001). Stein rule restricted regression estimator in a linear model with non spherical disturbances. Communication in Statistics-Theory and Methods, 30, 1, 55-68.

Daniel, C., Wood, F. S. (1980). Fitting equations to data. Computer analysis of multifactor data, Second Edition, with the assistance of JohnW. Gorman,Wiley, New York.

Eldum. H. and Zahri M., (2013). Relaxation Method For Two Stages Ridge Regression Estimator. International Journal of Pure and Applied Mathematics, Volume 85, 4, 653-667.

Firinguetti, L. (1989). A simulation study of ridge regression estimators with autocorrelated errors. Communications in Statistics – Simulation and Computation, 18, 2, 673-702.

Gorman, J. W., Toman, R. J. (1966). Selection of variables for fitting equations to data. Technometrics, 8, 27–51.

Gosling, B. J., Hsu, J. J. and Puterman, M. L. (1982). Ridge Estimation in Regression Problems with Autocorrelated Errors. Working paper no. 810. Faculty of Commerce and Business Administration. Vancouver: University of British Columbia.

Güler, H. and Kaçıranlar, S. (2009). A comparison of mixed and ridge estimators of linear models. Commun. Stat. Simul. Comput., 38, 2, 368–401.

Hald, A. (1952). Statistical theory with engineering applications, Wiley, New York.

Hamaker, H.C. (1962). On multiple regression analysis. Statistica Neerlandica, 16, 31-56.

Hoerl, A. E., Kennard, R. W. (1970a). Ridge regression: Biased estimation for nonorthogonal problems. Technometrics, 12, 55–67.

Hoerl, A.E.; Kennard, R.W. (1970b). Ridge regression: Application for nonorthogonal problems. Technometrics, 12, 69–82.

Hoerl A.E., R.W. Kennard, K.F. Baldwin, (1975). Ridge regression: Some simulations. Communications in Statistics – Simulation and Computation, 4, 105-123.

James, W., Stein, C. (1961). Estimation with quadratic loss. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. Berkeley, University of California Press. 361-379.

Judge, G. C., Griffiths, W. E., Hill, R. C., Lütkepohl, H., Lee, T. C. (1985). The Theory and Practice of Econometrics. 2nd ed. New York: John Wiley & Sons.

Kaçıranlar S., (2003). Liu estimator in the general linear regression model. Journal of Applied Statistical Science, 13, 229-234.

Kaçıranlar S., Sakallıoğlu, S., Akdeniz F., Styan, G.P.H., Werner, H.J., (1999). A new biased estimator in linear regression and detailed analysis of the widely-analysed dataset on Portland cement, Sankhya B, 61 , 443-459.

Kibria, B. M. G. (2003). Performance of some new ridge regression estimators. Commun. Stat. Simul. Comput., 32, 2, 419–435.

Lawless, J. F., Wang, P. (1976). A simulation study of ridge and other regression estimators. Communications in Statistics-Theory and Methods, 4, 307–323.

Liu, K. (1993). A New Class of Biased Estimate in Linear Regression. Communications in Statistics-Theory and Methods, 22, 2, 393-402.

Liu, K. (2003). Using Liu-type estimator to combat collinearity. Communications in Statistics—Theory and Methods, 32:1009–1020.

McDonald, G. C. and Galarneau, D. I. (1975). A monte carlo evaluation of some ridge-type estimators. J. Am. Stat. Assoc., 20, 407–416.

Muniz, G., Kibria, B. M. G. (2009). On Some Ridge Regression Estimators: An Empirical Comparisons. Communications in Statistics - Simulation and Computation, 3, 621- 630.

Nomura, M. (1988). On the almost unbiased ridge regression estimator. Communications in Statistics- Simulation and Computation, 17, 729–743.

Özkale, M. R. (2008). A jackknifed ridge estimator in the linear regression model with heteroscedastic or correlated errors. Statistics and Probability Letters, 78, 3159-3169.

Özkale, M. R. (2014). Monte Carlo Simulation Study of Biased Estimators in the Linear Regression Models with Correlated or Heteroscedastic Errors. Communications in Statistics- Simulation and Computation, 43, 5, 1143-1186.

Piegel, G., Redgate, T. (1998). A mixture experiment analysis of the Hald cement data. The American Statistician, 1, 23–30.

Prais, S. J. and Winsten, C. B. (1954). Trend Estimators and Serial Correlation. Chicago: Cowles Commission.

Sakallıoğlu, S., Kacıranlar, S., Akdeniz, F. (2001). Mean Squared Error Comparisons of Some Biased Regression Estimators. Communications in Statistics - Theory and Methods, 2, 347-361.

Sakallıoğlu, S., Kacıranlar, S. (2008). A new biased estimator based on ridge estimation. Statistical Papers, 49:669–689.

Stein, C. (1956). Inadmissibility of usual estimator for the mean of a multivariate Normal distribution. Proceedings of the Third Berkeley Symposium on Mathmatical Statistics and Probability. Berkeley, University of California Press. 197-206.

Stein, C. (1975). Estimation of a covariance matrix. Rietz lecture, 39th Annual Meeting IMS. Atlanta, Georgia.

Şiray, G. U., Kaçıranlar, S. and Sakallıoğlu, S. (2014). $r - k$ Class estimator in the linear regression model with correlated errors. Statistical Papers, 55, 393-407.

Trenkler, G. (1980). Generalized Mean Square Error Comparisons of Biased Regression Estimators. Communications in Statistics-Theory and Methods, 12, 1247–1259.

Trenkler, G. (1984). On the performance of biased estimators in the linear regression model with correlated or heteroscedastic errors, Journal of Econometrics, 25, 179-190.

Woods, H., Steinour, H. H., Starke, H. R. (1932). Effect of composition of Portland cement on heat evolved during hardening. Indust. Eng. Chem. 24, 1207–1241.

## Appendices

**Table 1**

For $\sigma = 0.1$

| $\gamma$ | n | $\rho$ | OLS | TS | ORR | TR | SLS | STS | LIU | TL |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.7** | **20** | 0.1 | 0.00400 | 0.00390 | 0.00390 | 0.00380 | 0.00400 | 0.00390 | 0.00390 | **0.00380** |
| | | 0.3 | 0.00580 | 0.00380 | 0.00560 | 0.00380 | 0.00570 | 0.00380 | 0.00540 | **0.00370** |
| | | 0.5 | 0.01230 | 0.00370 | 0.01200 | 0.00360 | 0.01230 | 0.00370 | 0.01090 | **0.00340** |
| | | 0.7 | 0.04110 | 0.00330 | 0.04020 | 0.00320 | 0.04100 | 0.00330 | 0.03440 | **0.00290** |
| | | 0.9 | 0.27280 | 0.00250 | 0.26660 | 0.00240 | 0.26860 | 0.00250 | 0.22090 | **0.00220** |
| | **60** | 0.1 | 0.00130 | 0.00120 | 0.00130 | 0.00120 | 0.00130 | 0.00120 | 0.00130 | **0.00120** |
| | | 0.3 | 0.00190 | 0.00120 | 0.00190 | 0.00120 | 0.00190 | 0.00120 | 0.00180 | **0.00110** |
| | | 0.5 | 0.00390 | 0.00100 | 0.00390 | 0.0010 | 0.00390 | 0.00100 | 0.00370 | **0.00095** |
| | | 0.7 | 0.01380 | 0.00074 | 0.01370 | 0.00073 | 0.01380 | 0.00073 | 0.01290 | **0.00072** |
| | | 0.9 | 0.16540 | 0.00052 | 0.16420 | 0.00052 | 0.16170 | 0.00052 | 0.15430 | **0.00051** |
| **0.8** | **20** | 0.1 | 0.00550 | 0.00530 | 0.00530 | 0.00510 | 0.00550 | 0.00530 | 0.00450 | **0.00430** |
| | | 0.3 | 0.00770 | 0.00520 | 0.00750 | 0.00510 | 0.00770 | 0.00520 | 0.00630 | **0.00430** |
| | | 0.5 | 0.01650 | 0.00520 | 0.01600 | 0.00500 | 0.01650 | 0.00520 | 0.01310 | **0.00410** |
| | | 0.7 | 0.05600 | 0.00470 | 0.05430 | 0.00450 | 0.05590 | 0.00470 | 0.04280 | **0.00360** |
| | | 0.9 | 0.38170 | 0.00360 | 0.36940 | 0.00350 | 0.37670 | 0.00360 | 0.28400 | **0.00290** |
| | **60** | 0.1 | 0.00180 | 0.00170 | 0.00180 | 0.00170 | 0.00180 | 0.00170 | 0.00170 | **0.00160** |
| | | 0.3 | 0.00260 | 0.00160 | 0.00260 | 0.00160 | 0.00260 | 0.00160 | 0.00240 | **0.00150** |
| | | 0.5 | 0.00540 | 0.00130 | 0.00530 | 0.00130 | 0.00540 | 0.00130 | 0.00490 | **0.00120** |
| | | 0.7 | 0.01890 | 0.00100 | 0.01870 | 0.00100 | 0.01890 | 0.00100 | 0.01710 | **0.00095** |
| | | 0.9 | 0.22680 | 0.00072 | 0.22440 | 0.00071 | 0.22200 | 0.00070 | 0.20470 | **0.00067** |
| **0.9** | **20** | 0.1 | 0.01000 | 0.00980 | 0.00930 | 0.00930 | 0.01000 | 0.00980 | 0.00600 | **0.00580** |
| | | 0.3 | 0.01400 | 0.00970 | 0.01310 | 0.00920 | 0.01400 | 0.00970 | 0.00860 | **0.00580** |
| | | 0.5 | 0.02980 | 0.00960 | 0.02800 | 0.00910 | 0.02980 | 0.00960 | 0.01830 | **0.00570** |
| | | 0.7 | 0.10260 | 0.00910 | 0.09650 | 0.00850 | 0.10240 | 0.00910 | 0.06190 | **0.00520** |
| | | 0.9 | 0.71580 | 0.00720 | 0.67290 | 0.00680 | 0.70810 | 0.00720 | 0.42210 | **0.00430** |
| | **60** | 0.1 | 0.00330 | 0.00310 | 0.00320 | 0.00310 | 0.00330 | 0.00310 | 0.00280 | **0.00260** |
| | | 0.3 | 0.00480 | 0.00300 | 0.00470 | 0.00290 | 0.00480 | 0.00300 | 0.00400 | **0.00250** |
| | | 0.5 | 0.01000 | 0.00250 | 0.00980 | 0.00240 | 0.01000 | 0.00250 | 0.00830 | **0.00210** |
| | | 0.7 | 0.03500 | 0.00190 | 0.03430 | 0.00180 | 0.03500 | 0.00190 | 0.02890 | **0.00160** |
| | | 0.9 | 0.42220 | 0.00130 | 0.41350 | 0.00130 | 0.41440 | 0.00130 | 0.34690 | **0.00120** |

**Table 2**

For $\sigma = 0.5$

| $\gamma$ | n | $\rho$ | OLS | TS | ORR | TR | SLS | STS | LIU | TL |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.7** | **20** | 0.1 | 0.10080 | 0.09690 | 0.09830 | 0.09450 | 0.10000 | 0.09620 | 0.08060 | **0.07770** |
| | | 0.3 | 0.14380 | 0.09600 | 0.14030 | 0.09370 | 0.14220 | 0.09520 | 0.11590 | **0.07700** |
| | | 0.5 | 0.30720 | 0.09310 | 0.30010 | 0.09080 | 0.30100 | 0.09230 | 0.24840 | **0.07400** |
| | | 0.7 | 1.02760 | 0.08140 | 1.00410 | 0.07940 | 0.97840 | 0.08050 | 0.83050 | **0.06480** |
| | | 0.9 | 6.82120 | 0.06200 | 6.66410 | 0.06060 | 6.20290 | 0.06080 | 5.49090 | **0.05040** |
| | **60** | 0.1 | 0.03260 | 0.03100 | 0.03230 | 0.03080 | 0.03250 | 0.03090 | 0.03040 | **0.02890** |
| | | 0.3 | 0.04700 | 0.02920 | 0.04670 | 0.02900 | 0.04680 | 0.02910 | 0.04380 | **0.02720** |
| | | 0.5 | 0.09780 | 0.02450 | 0.09700 | 0.02430 | 0.09690 | 0.02440 | 0.09110 | **0.02290** |
| | | 0.7 | 0.34450 | 0.01860 | 0.34200 | 0.01850 | 0.33390 | 0.01850 | 0.32070 | **0.01750** |
| | | 0.9 | 4.13540 | 0.01310 | 4.10560 | 0.01300 | 3.15710 | 0.01300 | 3.85520 | **0.01240** |
| **0.8** | **20** | 0.1 | 0.13640 | 0.13150 | 0.13160 | 0.12690 | 0.13550 | 0.13060 | 0.09930 | **0.09590** |
| | | 0.3 | 0.19300 | 0.13120 | 0.18650 | 0.12660 | 0.19130 | 0.13020 | 0.14250 | **0.09580** |
| | | 0.5 | 0.41250 | 0.13040 | 0.39910 | 0.12560 | 0.40550 | 0.12920 | 0.30690 | **0.09370** |
| | | 0.7 | 1.40120 | 0.11730 | 1.35620 | 0.11300 | 1.34180 | 0.11600 | 1.04390 | **0.08420** |
| | | 0.9 | 9.54210 | 0.09080 | 9.23310 | 0.08780 | 8.72030 | 0.08920 | 7.06980 | **0.06710** |
| | **60** | 0.1 | 0.04480 | 0.04260 | 0.04430 | 0.04210 | 0.04460 | 0.04250 | 0.04050 | **0.03850** |
| | | 0.3 | 0.06470 | 0.04010 | 0.06400 | 0.03970 | 0.06440 | 0.04000 | 0.05840 | **0.03630** |
| | | 0.5 | 0.13460 | 0.03370 | 0.13320 | 0.03330 | 0.13360 | 0.03360 | 0.12160 | **0.03060** |
| | | 0.7 | 0.47360 | 0.02550 | 0.46850 | 0.02530 | 0.46060 | 0.02540 | 0.42720 | **0.02350** |
| | | 0.9 | 5.67070 | 0.01790 | 5.61040 | 0.01780 | 4.35760 | 0.01780 | 5.11600 | **0.01670** |
| **0.9** | **20** | 0.1 | 0.24910 | 0.24590 | 0.23260 | 0.22860 | 0.24770 | 0.24390 | 0.14100 | **0.13600** |
| | | 0.3 | 0.34920 | 0.24140 | 0.32710 | 0.22550 | 0.34650 | 0.23970 | 0.20260 | **0.13600** |
| | | 0.5 | 0.74490 | 0.24060 | 0.69940 | 0.22480 | 0.73400 | 0.23910 | 0.43910 | **0.13520** |
| | | 0.7 | 2.56580 | 0.22850 | 2.41240 | 0.21220 | 2.47080 | 0.22620 | 1.52110 | **0.12460** |
| | | 0.9 | 17.8960 | 0.18060 | 16.8203 | 0.16880 | 16.44270 | 0.17760 | 10.5212 | **0.10280** |
| | **60** | 0.1 | 0.08270 | 0.07870 | 0.08110 | 0.07710 | 0.08260 | 0.07850 | 0.06830 | **0.06490** |
| | | 0.3 | 0.11960 | 0.07410 | 0.11720 | 0.07260 | 0.11920 | 0.07390 | 0.09880 | **0.06130** |
| | | 0.5 | 0.24890 | 0.06210 | 0.24390 | 0.06090 | 0.24730 | 0.06190 | 0.20550 | **0.05200** |
| | | 0.7 | 0.87490 | 0.04690 | 0.85710 | 0.04610 | 0.85400 | 0.04680 | 0.72050 | **0.04010** |
| | | 0.9 | 10.5552 | 0.03300 | 10.3373 | 0.03250 | 8.23070 | 0.03280 | 8.67260 | **0.02890** |

**Table 3**

For $\sigma = 1$

| $\gamma$ | n | $\rho$ | OLS | TS | ORR | TR | SLS | STS | LIU | TL |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.7** | **20** | 0.1 | 0.40310 | 0.38780 | 0.39300 | 0.37820 | 0.39110 | 0.37620 | 0.32020 | **0.30850** |
| | | 0.3 | 0.57500 | 0.38410 | 0.56130 | 0.37470 | 0.55250 | 0.37220 | 0.46120 | **0.30600** |
| | | 0.5 | 1.22890 | 0.37260 | 1.20050 | 0.36310 | 1.15100 | 0.35940 | 0.99070 | **0.29440** |
| | | 0.7 | 4.11050 | 0.32580 | 4.01650 | 0.31760 | 3.70320 | 0.31120 | 3.31830 | **0.25820** |
| | | 0.9 | 27.2847 | 0.24780 | 26.6562 | 0.24220 | 24.2084 | 0.23070 | 21.9590 | **0.20090** |
| | **60** | 0.1 | 0.13030 | 0.12390 | 0.12940 | 0.12300 | 0.12870 | 0.12250 | 0.12140 | **0.11530** |
| | | 0.3 | 0.18810 | 0.11680 | 0.18670 | 0.11590 | 0.18490 | 0.11540 | 0.17520 | **0.10870** |
| | | 0.5 | 0.39110 | 0.09800 | 0.38820 | 0.09740 | 0.37810 | 0.09680 | 0.36410 | **0.09170** |
| | | 0.7 | 1.37810 | 0.07430 | 1.36790 | 0.07380 | 1.23580 | 0.07320 | 1.28240 | **0.07000** |
| | | 0.9 | 16.5416 | 0.05230 | 16.4223 | 0.05200 | 11.7350 | 0.05110 | 15.4206 | **0.04970** |
| **0.8** | **20** | 0.1 | 0.54570 | 0.52600 | 0.52630 | 0.50750 | 0.53110 | 0.51180 | 0.39550 | **0.38210** |
| | | 0.3 | 0.77220 | 0.52470 | 0.74610 | 0.50640 | 0.74550 | 0.50970 | 0.56820 | **0.38170** |
| | | 0.5 | 1.65020 | 0.52140 | 1.59640 | 0.50230 | 1.55590 | 0.50420 | 1.22510 | **0.37380** |
| | | 0.7 | 5.60480 | 0.46930 | 5.42490 | 0.45200 | 5.07920 | 0.44930 | 4.17220 | **0.33610** |
| | | 0.9 | 38.1682 | 0.36330 | 36.9321 | 0.35120 | 33.9684 | 0.33890 | 28.2746 | **0.26780** |
| | **60** | 0.1 | 0.17910 | 0.17030 | 0.17720 | 0.16850 | 0.17710 | 0.16860 | 0.16170 | **0.15370** |
| | | 0.3 | 0.25870 | 0.16050 | 0.25600 | 0.15890 | 0.25490 | 0.15890 | 0.23370 | **0.14510** |
| | | 0.5 | 0.53840 | 0.13470 | 0.53270 | 0.13340 | 0.52260 | 0.13320 | 0.48620 | **0.12250** |
| | | 0.7 | 1.89430 | 0.10200 | 1.87410 | 0.10110 | 1.71750 | 0.10070 | 1.70880 | **0.09380** |
| | | 0.9 | 22.6829 | 0.07180 | 22.4416 | 0.07120 | 15.9184 | 0.07020 | 20.4639 | **0.06680** |
| **0.9** | **20** | 0.1 | 0.99660 | 0.98340 | 0.93020 | 0.91460 | 0.97330 | 0.95370 | 0.56280 | **0.54610** |
| | | 0.3 | 1.39670 | 0.96540 | 1.30830 | 0.90200 | 1.35480 | 0.94040 | 0.80890 | **0.54470** |
| | | 0.5 | 2.97940 | 0.96230 | 2.79760 | 0.89900 | 2.82820 | 0.93940 | 1.75390 | **0.54000** |
| | | 0.7 | 10.2633 | 0.91410 | 9.64970 | 0.84890 | 9.35990 | 0.87800 | 6.08090 | **0.49760** |
| | | 0.9 | 71.5838 | 0.72250 | 67.2808 | 0.67510 | 63.8998 | 0.67640 | 42.0796 | **0.41080** |
| | **60** | 0.1 | 0.33100 | 0.31480 | 0.32430 | 0.30840 | 0.32790 | 0.31200 | 0.27320 | **0.25950** |
| | | 0.3 | 0.47830 | 0.29640 | 0.46860 | 0.29050 | 0.47220 | 0.29370 | 0.39500 | **0.24520** |
| | | 0.5 | 0.99570 | 0.24820 | 0.97560 | 0.24360 | 0.97040 | 0.24590 | 0.82190 | **0.20800** |
| | | 0.7 | 3.49970 | 0.18760 | 3.42830 | 0.18450 | 3.20820 | 0.18540 | 2.88210 | **0.16040** |
| | | 0.9 | 42.2209 | 0.13180 | 41.3493 | 0.13000 | 29.5067 | 0.12920 | 34.6903 | **0.11540** |

**Table 4**

For $\sigma = 4$

| $\gamma$ | n | $\rho$ | OLS | TS | ORR | TR | SLS | STS | LIU | TL |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.7** | **20** | 0.1 | 6.44980 | 6.20420 | 6.28790 | 6.05030 | 5.07820 | **4.86230** | 5.11030 | 4.92450 |
| | | 0.3 | 9.20050 | 6.14540 | 8.98090 | 5.99500 | 7.36510 | **4.77360** | 7.36780 | 4.88810 |
| | | 0.5 | 19.6632 | 5.96110 | 19.2077 | 5.80970 | 16.0776 | **4.45090** | 15.8372 | 4.70450 |
| | | 0.7 | 65.7674 | 5.21270 | 64.2639 | 5.08190 | 55.5152 | **3.51680** | 53.0740 | 4.12640 |
| | | 0.9 | 436.555 | 3.96500 | 426.498 | 3.87570 | 382.422 | **2.17510** | 351.316 | 3.21080 |
| | **60** | 0.1 | 2.08500 | 1.98280 | 2.06970 | 1.96810 | 1.78630 | **1.70550** | 1.94080 | 1.84440 |
| | | 0.3 | 3.00990 | 1.86850 | 2.98770 | 1.85480 | 2.50190 | **1.60480** | 2.80180 | 1.73950 |
| | | 0.5 | 6.25700 | 1.56860 | 6.21090 | 1.55770 | 4.88030 | **1.32880** | 5.82370 | 1.46650 |
| | | 0.7 | 22.0498 | 1.18850 | 21.8867 | 1.18120 | 16.0425 | **0.96750** | 20.5168 | 1.11950 |
| | | 0.9 | 264.665 | 0.83710 | 262.757 | 0.83280 | 181.628 | **0.61470** | 246.727 | 0.79570 |
| **0.8** | **20** | 0.1 | 8.73120 | 8.41550 | 8.42020 | 8.11950 | 6.31950 | **6.10450** | 6.93530 | 6.65150 |
| | | 0.3 | 12.3546 | 8.39530 | 11.9375 | 8.10270 | 9.08270 | **6.10050** | 9.93780 | 6.58410 |
| | | 0.5 | 26.4031 | 8.34310 | 25.5421 | 8.03710 | 19.5893 | **5.97500** | 21.7634 | 6.28810 |
| | | 0.7 | 89.6762 | 7.50850 | 86.7975 | 7.23230 | 75.9630 | **5.12120** | 66.7379 | 5.37400 |
| | | 0.9 | 610.691 | 5.81240 | 590.911 | 5.61840 | 536.198 | **3.17970** | 452.363 | 4.28200 |
| | **60** | 0.1 | 2.86490 | 2.72500 | 2.83470 | 2.69600 | 2.48270 | **2.36980** | 2.58670 | 2.45830 |
| | | 0.3 | 4.13940 | 2.56860 | 4.09580 | 2.54180 | 3.47990 | **2.23060** | 3.73800 | 2.32120 |
| | | 0.5 | 8.61410 | 2.15530 | 8.52330 | 2.13420 | 6.79570 | **1.85200** | 7.77750 | 1.96030 |
| | | 0.7 | 30.3085 | 1.63180 | 29.9858 | 1.61760 | 22.0855 | **1.34970** | 27.3386 | 1.50020 |
| | | 0.9 | 362.926 | 1.14830 | 359.064 | 1.13990 | 244.336 | **0.83460** | 327.421 | 1.06930 |
| **0.9** | **20** | 0.1 | 15.9454 | 15.7347 | 14.8830 | 14.6333 | 8.99870 | **8.73250** | 12.8012 | 12.3038 |
| | | 0.3 | 22.3479 | 15.4469 | 20.9331 | 14.4320 | 12.9362 | **8.70930** | 18.0908 | 12.2405 |
| | | 0.5 | 47.6711 | 15.3974 | 44.7620 | 14.3837 | 28.0504 | **8.63590** | 39.3592 | 12.0052 |
| | | 0.7 | 164.213 | 14.6261 | 154.393 | 13.5825 | 97.2736 | **7.95820** | 139.464 | 10.1408 |
| | | 0.9 | 1145.30 | 11.6000 | 1076.50 | 10.8000 | 673.200 | **6.30000** | 1007.40 | 6.60000 |
| | **60** | 0.1 | 5.29570 | 5.03690 | 5.18890 | 4.93460 | 4.37030 | **4.15080** | 4.65920 | 4.44610 |
| | | 0.3 | 7.65220 | 4.74180 | 7.49830 | 4.64740 | 6.31850 | **3.92320** | 6.54520 | 4.18740 |
| | | 0.5 | 15.9306 | 3.97130 | 15.6098 | 3.89780 | 13.1498 | **3.32810** | 12.6967 | 3.46200 |
| | | 0.7 | 55.9950 | 3.00100 | 54.8532 | 2.95190 | 40.9063 | **2.52570** | 46.1132 | 2.56620 |
| | | 0.9 | 675.533 | 2.10890 | 661.588 | 2.07980 | 446.404 | **1.54280** | 555.045 | 1.84640 |

**Table 5**
For $\sigma = 9$

| $\gamma$ | n | $\rho$ | OLS | TS | ORR | TR | SLS | STS | LIU | TL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 32.6519 | 31.4087 | 31.8320 | 30.6295 | 23.6928 | **22.6089** | 25.8664 | 24.9259 |
| | | 0.3 | 46.5776 | 31.1112 | 45.4661 | 30.3497 | 35.2403 | **21.9844** | 37.2981 | 24.7447 |
| | **20** | 0.5 | 99.5449 | 30.1780 | 97.2390 | 29.4116 | 79.4956 | **20.0158** | 80.1728 | 23.8155 |
| | | 0.7 | 332.947 | 26.3891 | 325.335 | 25.7274 | 279.238 | **15.1105** | 268.681 | 20.8890 |
| | | 0.9 | 2210.10 | 20.1000 | 2159.10 | 19.6000 | 1934.10 | **11.3000** | 1778.50 | 16.3000 |
| **0.7** | | 0.1 | 10.5553 | 10.0381 | 10.4777 | 9.96350 | 7.62080 | **7.23800** | 9.82540 | 9.33740 |
| | | 0.3 | 15.2374 | 9.45920 | 15.1254 | 9.38980 | 10.9537 | **6.66130** | 14.1837 | 8.80600 |
| | **60** | 0.5 | 31.6760 | 7.94080 | 31.4428 | 7.88600 | 22.4672 | **5.27050** | 29.4819 | 7.42420 |
| | | 0.7 | 111.627 | 6.01690 | 110.801 | 5.98010 | 78.6438 | **3.53340** | 103.865 | 5.66770 |
| | | 0.9 | 1339.90 | 4.20000 | 1330.20 | 4.20000 | 918.300 | **3.90000** | 1249.10 | 4.00000 |
| | | 0.1 | 44.2016 | 42.6034 | 42.6273 | 41.1049 | 32.1547 | **30.7202** | 31.9894 | 30.9010 |
| | | 0.3 | 62.5453 | 42.5013 | 60.4340 | 41.0199 | 47.5176 | **30.2846** | 45.9803 | 30.8828 |
| | **20** | 0.5 | 133.665 | 42.2367 | 129.306 | 40.6881 | 107.076 | **28.2925** | 99.1680 | 30.2476 |
| | | 0.7 | 453.986 | 38.0119 | 439.411 | 36.6134 | 382.370 | **22.1694** | 337.854 | 27.2052 |
| | | 0.9 | 3091.60 | 29.4000 | 2991.50 | 28.4000 | 2711.90 | **15.6000** | 2290.10 | 21.7000 |
| **0.8** | | 0.1 | 14.5036 | 13.7951 | 14.3506 | 13.6484 | 10.5212 | **9.97150** | 13.0951 | 12.4447 |
| | | 0.3 | 20.9558 | 13.0038 | 20.7351 | 12.8678 | 15.0803 | **9.18420** | 18.9235 | 11.7507 |
| | **60** | 0.5 | 43.6090 | 10.9113 | 43.1492 | 10.8046 | 31.1998 | **7.31510** | 39.3732 | 9.92420 |
| | | 0.7 | 153.436 | 8.2608 | 151.803 | 8.18930 | 108.307 | **4.77770** | 138.401 | 7.59460 |
| | | 0.9 | 1837.30 | 5.80000 | 1817.80 | 5.80000 | 1234.60 | **0.00430** | 1657.60 | 0.00540 |
| | | 0.1 | 80.7234 | 79.6567 | 75.3449 | 74.0813 | 45.5531 | **44.2079** | 58.9509 | 56.3124 |
| | | 0.3 | 113.136 | 78.1999 | 105.974 | 73.0620 | 65.4888 | **44.0888** | 86.3287 | 56.0249 |
| | **20** | 0.5 | 241.335 | 77.9496 | 226.607 | 72.8176 | 142.001 | **43.7186** | 193.687 | 53.8833 |
| | | 0.7 | 831.330 | 74.0447 | 781.614 | 68.7613 | 492.439 | **40.2881** | 701.149 | 43.3820 |
| | | 0.9 | 5798.30 | 58.5000 | 5449.70 | 54.7000 | 5095.00 | **29.4000** | 3408.20 | 33.3000 |
| **0.9** | | 0.1 | 26.8097 | 25.4994 | 26.269 | 24.9812 | 19.6643 | **18.6580** | 22.1245 | 21.0134 |
| | | 0.3 | 38.7390 | 24.0052 | 37.9601 | 23.5276 | 28.2216 | **17.3363** | 31.9868 | 19.8613 |
| | **60** | 0.5 | 80.6489 | 20.1048 | 79.0248 | 19.7324 | 57.4337 | **13.5320** | 66.5715 | 16.8483 |
| | | 0.7 | 283.474 | 15.1926 | 277.694 | 14.9442 | 198.408 | **8.69680** | 233.448 | 12.9914 |
| | | 0.9 | 3419.90 | 10.7000 | 3349.30 | 10.5000 | 2252.30 | **5.50000** | 2809.90 | 9.30000 |

**Table 6**

For $\sigma = 20$

| $\gamma$ | n | $\rho$ | OLS | TS | ORR | TR | SLS | STS | LIU | TL |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.7** | **20** | 0.1 | 161.243 | 155.104 | 157.194 | 151.256 | 114.990 | **109.370** | 127.728 | 123.085 |
| | | 0.3 | 230.012 | 153.635 | 224.524 | 149.875 | 171.387 | **105.738** | 184.189 | 122.196 |
| | | 0.5 | 491.580 | 149.026 | 480.192 | 145.242 | 390.223 | **95.2341** | 395.914 | 117.607 |
| | | 0.7 | 1644.20 | 130.300 | 1606.60 | 127.000 | 1377.10 | **70.4000** | 1326.80 | 103.200 |
| | | 0.9 | 109140 | 99.0000 | 10662.0 | 97.0000 | 9549.00 | **60.0000** | 8783.00 | 80.0000 |
| | **60** | 0.1 | 52.1250 | 49.5707 | 51.7417 | 49.2026 | 34.9243 | **32.7802** | 48.5204 | 46.1108 |
| | | 0.3 | 75.2466 | 46.7122 | 74.6933 | 46.3692 | 51.2379 | **29.6174** | 70.0422 | 43.4865 |
| | | 0.5 | 156.424 | 39.2138 | 155.272 | 38.9433 | 107.975 | **22.3779** | 145.588 | 36.6631 |
| | | 0.7 | 551.244 | 29.7132 | 547.167 | 29.5313 | 385.467 | **15.5825** | 512.915 | 27.9886 |
| | | 0.9 | 6616.60 | 20.9000 | 6568.90 | 20.8000 | 4535.40 | **11.6000** | 6168.20 | 19.9000 |
| **0.8** | **20** | 0.1 | 218.279 | 210.387 | 210.505 | 202.987 | 156.396 | **148.725** | 157.967 | 152.593 |
| | | 0.3 | 308.865 | 209.883 | 298.439 | 202.567 | 231.055 | **144.947** | 227.064 | 152.507 |
| | | 0.5 | 660.078 | 208.576 | 638.552 | 200.928 | 525.454 | **134.669** | 489.716 | 149.371 |
| | | 0.7 | 2241.90 | 187.700 | 2169.90 | 180.800 | 1883.70 | **103.500** | 1668.40 | 134.300 |
| | | 0.9 | 152670 | 145.000 | 14773.0 | 140.000 | 13389.0 | **87.0000** | 11309.0 | 107.000 |
| | **60** | 0.1 | 71.6226 | 68.1240 | 70.8673 | 67.3994 | 48.0458 | **45.2322** | 64.6670 | 61.4549 |
| | | 0.3 | 103.485 | 64.2161 | 102.395 | 63.5444 | 71.0831 | **40.9048** | 93.4495 | 58.0275 |
| | | 0.5 | 215.353 | 53.8831 | 213.082 | 53.3562 | 149.000 | **30.9069** | 194.434 | 49.0085 |
| | | 0.7 | 757.713 | 40.7939 | 749.645 | 40.4409 | 528.454 | **21.2984** | 683.460 | 37.5045 |
| | | 0.9 | 9073.20 | 28.7000 | 8976.60 | 28.5000 | 6097.90 | **17.5000** | 8185.50 | 26.7000 |
| **0.9** | **20** | 0.1 | 398.634 | 393.400 | 372.073 | 365.800 | 224.949 | **218.311** | 286.237 | 271.932 |
| | | 0.3 | 558.697 | 386.172 | 523.328 | 360.800 | 323.402 | **217.719** | 418.647 | 267.845 |
| | | 0.5 | 1191.80 | 384.936 | 1119.00 | 359.592 | 701.200 | **215.900** | 950.700 | 257.000 |
| | | 0.7 | 4105.30 | 365.700 | 3859.80 | 339.600 | 2431.80 | **199.000** | 3456.70 | 206.300 |
| | | 0.9 | 286340 | 289.000 | 26912.0 | 270.000 | 16831.0 | **164.000** | 25154.0 | 172.000 |
| | **60** | 0.1 | 132.393 | 125.922 | 129.723 | 123.364 | 89.0723 | **83.9173** | 109.256 | 103.769 |
| | | 0.3 | 191.303 | 118.544 | 187.457 | 116.185 | 130.846 | **75.7438** | 157.958 | 98.0806 |
| | | 0.5 | 398.266 | 99.2828 | 390.246 | 97.4439 | 276.051 | **57.2628** | 328.749 | 83.2014 |
| | | 0.7 | 1399.90 | 75.0000 | 1371.30 | 73.8000 | 975.000 | **39.5000** | 1152.80 | 64.2000 |
| | | 0.9 | 168880 | 53.0000 | 16540.0 | 52.0000 | 11117.0 | **113.000** | 13876.0 | 46.0000 |

**Table 10**

| | $\hat{\beta}_k$ | | | $\hat{\beta}_{TR}$ | | |
|---|---|---|---|---|---|---|
| $k$ | $Var$ | $Bias^2$ | $mse$ | $Var$ | $Bias^2$ | $mse$ |
| 0 | 4912.100 | 0.000000 | 4912.100 | 4647.422 | 0.000000 | 4647.422 |
| **0.01** | **57.97370** | **3096.800** | **3154.774** | **59.44948** | **2321.100** | **2380.549** |
| 0.02 | 16.25220 | 3462.500 | 3478.752 | 16.74240 | 2607.500 | 2624.242 |
| 0.03 | 7.542900 | 3599.000 | 3606.543 | 7.781771 | 2714.900 | 2722.682 |
| 0.04 | 4.354700 | 3670.200 | 3674.555 | 4.495065 | 2771.100 | 2775.595 |
| 0.05 | 2.843200 | 3714.000 | 3716.843 | 2.935167 | 2805.600 | 2808.535 |
| 0.06 | 2.009700 | 3743.600 | 3745.610 | 2.074485 | 2829.000 | 2831.074 |
| 0.07 | 1.502000 | 3765.000 | 3766.502 | 1.549900 | 2845.900 | 2847.450 |
| 0.08 | 1.170000 | 3781.100 | 3782.270 | 1.206676 | 2858.600 | 2859.807 |
| 0.09 | 0.941000 | 3793.700 | 3794.641 | 0.969974 | 2868.600 | 2869.570 |
| 0.1 | 0.776500 | 3803.900 | 3804.677 | 0.799897 | 2876.700 | 2877.500 |
| 0.2 | 0.245000 | 3850.100 | 3850.345 | 0.250166 | 2913.200 | 2913.450 |
| 0.3 | 0.145300 | 3865.700 | 3865.845 | 0.146813 | 2925.600 | 2925.747 |
| 0.4 | 0.110100 | 3873.600 | 3873.710 | 0.110481 | 2931.800 | 2931.910 |
| 0.5 | 0.093700 | 3878.300 | 3878.394 | 0.093552 | 2935.500 | 2935.594 |
| 0.6 | 0.084800 | 3881.500 | 3881.585 | 0.084247 | 2938.000 | 2938.084 |
| 0.7 | 0.079300 | 3883.800 | 3883.879 | 0.078604 | 2939.900 | 2939.979 |
| 0.8 | 0.075800 | 3885.500 | 3885.576 | 0.074941 | 2941.200 | 2941.275 |
| 0.9 | 0.073300 | 3886.900 | 3886.973 | 0.072367 | 2942.300 | 2942.372 |
| 1 | 0.071500 | 3888.000 | 3888.072 | 0.070486 | 2943.200 | 2943.270 |

**Table 11**

| | $\hat{\beta}_d$ | | | $\hat{\beta}_{TL}$ | | |
|---|---|---|---|---|---|---|
| $d$ | $Var$ | $Bias^2$ | $mse$ | $Var$ | $Bias^2$ | $mse$ |
| 0 | 0.07150 | 3888.00 | 3888.072 | 0.070486 | 2943.200 | 2943.270 |
| 0.01 | 0.68090 | 3810.60 | 3811.281 | 0.652193 | 2884.600 | 2885.252 |
| 0.02 | 2.27030 | 3734.00 | 3736.270 | 2.161108 | 2826.700 | 2828.861 |
| 0.03 | 4.83970 | 3658.20 | 3663.040 | 4.597032 | 2769.300 | 2773.897 |
| 0.04 | 8.38920 | 3583.20 | 3591.589 | 7.960065 | 2712.500 | 2720.460 |
| 0.05 | 12.9187 | 3508.90 | 3521.819 | 12.25030 | 2656.200 | 2668.450 |
| 0.06 | 18.4282 | 3435.40 | 3453.828 | 17.46755 | 2600.600 | 2618.068 |
| 0.07 | 24.9177 | 3362.70 | 3387.618 | 23.61201 | 2545.600 | 2569.212 |
| 0.08 | 32.3873 | 3290.80 | 3323.187 | 30.68358 | 2491.100 | 2521.784 |
| 0.09 | 40.8368 | 3219.70 | 3260.537 | 38.68225 | 2437.300 | 2475.982 |
| 0.1 | 50.2664 | 3149.30 | 3199.566 | 47.60803 | 2384.000 | 2431.608 |
| 0.2 | 198.463 | 2488.30 | 2686.763 | 187.8570 | 1883.600 | **2071.457** |
| 0.3 | 444.663 | 1905.10 | 2349.763 | 420.8171 | 1442.200 | **1863.017** |
| **0.4** | **788.865** | **1399.70** | **2188.565** | **746.4901** | **1059.600** | **1806.090** |
| 0.5 | 1231.10 | 972.000 | 2203.100 | 1164.900 | 735.8000 | **1900.700** |
| 0.6 | 1771.30 | 622.080 | 2393.380 | 1675.923 | 470.9120 | **2146.835** |
| 0.7 | 2409.50 | 349.920 | 2759.420 | 2279.806 | 264.8880 | 2544.694 |
| 0.8 | 3145.70 | 155.520 | 3301.220 | 2976.251 | 117.7280 | 3093.979 |
| 0.9 | 3979.90 | 38.8800 | 4018.780 | 3765.555 | 29.43200 | 3794.987 |
| 1 | 4912.10 | 0.00000 | 4912.100 | 4647.422 | 0.000000 | 4647.422 |