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A NOTE ON M1 PARTITIONS OF *n*

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ABSTRACT

S.Ahlgren, Bringmann and Lovejoy [1] defined $M2spt(n)$ to be the number of smallest parts in the partitions of n without repeated odd parts and with smallest part even and Bringmann, Lovejoy and Osburn [4]derived the generating function for $M2spt(n)$. Hanumareddy and Manjusri [5] derived generating function for the number of smallest parts of partitions of n by using $r-$ partitions of n. In this chapter we defined $M1spt(n)$ as the number of smallest parts in the partitions of n without repeated even parts and with smallest part odd and also derive its generating function by using $r-M1partitions\ of n$. We also derive generating function for $M1spt(n)$.

Keywords: Partition, r-partition, M1Partition, Smallest part of the M1Partition.

Subject classification: 11P81 Elementary theory of Partitions**.**

1. Introduction:

Let $M1\xi(n)$ be denote the set of all $M1partitions$ of n with even numbers not repeated and smallest parts are odd numbers. Let $M1p(n)$ be the cardinality of $M1\xi(n)$, write $M1p_r(n)$ for the number of $r-M1partitions$ of n in $M1\xi(n)$ each consisting of exactly r parts, i.e $r-M1partitions$ of n in $M1\xi(n)$. Let $M1p(k,n)$ represent the number of $M1partitions$ of n in $M1\xi(n)$ using natural numbers at least as large as k only. Let the *partitions* in $M1\xi(n)$ be denoted by $M1partitions$.

Let $M1spt(n)$ be denotes the number of smallest parts including repetitions in all *partitions* of n in $M1\xi(n)$ and $sumM1spt(n)$ be denotes the sum of the smallest parts.

$M1m_s(\lambda)=$ number of smallest parts of λ in $M1\xi(n)$.

$M1spt(n)=\sum_{\lambda\in\xi(n)} M1m_s(\lambda)$

$M^1\xi(n)$ be denote set of all $M^1 partitions$ of n.

For example $M1\xi(7)$: $M1p(7)=11$ $M1spt(7)=28$

$\underline{7}$, $6+\underline{1}$, $4+\underline{3}$, $5+\underline{1}+\underline{1}$, $4+2+\underline{1}$, $3+3+\underline{1}$, $4+\underline{1}+\underline{1}+\underline{1}$, $3+2+\underline{1}+\underline{1}$, $3+\underline{1}+\underline{1}+\underline{1}+\underline{1}$, :
$2+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}$, $\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}$.

We observe that

1.1. The generating function for the number of $r-partitions$ of n with even numbers not repeated is

$$M^1p_r(n)=\frac{q^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}$$

1.2. The generating function for the number of $r-M1partitions$ of n with even numbers appears at most one time and smallest parts are odd numbers is

$$M1p_r(n)=\frac{q^r\left(-q,q^2\right)_{r-1}}{\left(q^2,q^2\right)_r} \qquad (1.1)$$

2. Generating function for $M1spt(n)$

The generating function for the number of smallest parts of all partitions of positive integer *n* is derived by G.E. Andrews. By utilizing $r-M1partitions$ of *n*, we propose a formula for finding the number of smallest parts of *n*.

2.1 Theorem:

$$M1spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}M^1p\left(2k-1,n-(2k-1)t\right)+\beta \quad \text{where} \quad \beta=\begin{cases}1 & \text{if } 2k-1\,|\,n\\ 0 & \text{otherwise}\end{cases}$$

Proof: Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r)=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k_1^{\alpha_l}\right)$,

$\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k_1^{\alpha_l}\right)\in M1\xi(n)$, $k_1=2k-1, k\in N$ be any $r-M1 partition$ of n with l distinct parts such that even parts not repeated and smallest parts are odd numbers

Case 1: Let $r>\alpha_l=t$ which implies $\lambda_{r-t}>k_1$. Subtract all $k_1's$, we get $n-tk_1=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}}\right),\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}}\right)\in M1\xi(n)$

Hence $n-tk_1=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t)-M^1 partition$ of $n-tk_1$ with l- 1 distinct parts and each part is greater than or equal to k_1+1. Here we get the number of $r-M1 partitions$ with smallest part k_1 that occurs exactly t times among all $r-M1 partitions$ of n is $M^1 p_{r-t}\left(k_1+1,n-tk_1\right)$.

Case 2: Let $r>\alpha_l>t$ which implies $\lambda_{r-t}=k_1$

Omit $k_1's$ from last t places, we get $n-tk_1=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k_1^{\alpha_l-t}\right)$, $\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k_1^{\alpha_l-t}\right)\in M1\xi(n)$. Hence $n-tk_1=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k_1^{\alpha_l-t}\right)$ is a $(r-t)-M1 partition$ of $n-tk_1$ with l distinct parts and the least part is k_1.

Now we get the number of $r-M1 partitions$ with smallest part k_1 that occurs more than t times among all $r-M1 partitions$ of n is $M^1 f_{r-t}\left(k_1,n-tk_1\right)$

Case 3: Let $r=\alpha_l=t$ which implies all parts in the *partition* are equal which are odd.

The number of *partitions* of n with equal parts in set $\left\{M1\xi(n),k_1\in 2N-1\right\}$ is equal to the number of divisors of $2n-1$. Since the number of divisors of $2n-1$ is $d(2n-1)$, the number of *partitions* of n with equal parts in set $\left\{M1\xi(n),k_1\in 2N-1\right\}$ is $d(2n-1)$ where $\beta=\begin{cases}1 & \text{if } k_1 \mid n\\ 0 & \text{otherwise}\end{cases}$

From cases (1), (2) and (3) we get $r-M1 partitions$ of n with smallest part k_1 that occurs t times is

$$M^1 f_{r-t}\left(k_1,n-tk_1\right)+M^1 p_{r-t}\left(k_1+1,n-tk_1\right)+\beta$$

$$=M^1 p_{r-t}\left(k_1,n-tk_1\right)+\beta \qquad \text{where} \quad \beta=\begin{cases}1 & \text{if } k_1 \mid n\\ 0 & \text{otherwise}\end{cases}$$

The number of smallest parts in $M1 partitions$ of n is

$$M1spt(n)=\sum_{k_1=1}^{\infty}\sum_{t=1}^{\infty}M^1p(k_1,n-tk_1)+\beta \quad \text{where} \quad \beta=\begin{cases}1 & \text{if } k_1 \mid n\\ 0 & \text{otherwise}\end{cases}$$

$$\Rightarrow M1spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}M^1p\left(2k-1,n-(2k-1)t\right)+\beta \quad \text{where} \quad \beta=\begin{cases}1 & \text{if } 2k-1 \mid n\\ 0 & \text{otherwise}\end{cases}$$

2.2. Theorem: $M^1p_r(2k+1,n)=M^1p_r(n-2kr)$

Proof: Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r),\lambda_i>2k \quad \forall i$, be any $r-M1partition$ of n such that even numbers not repeated and smallest parts are odd numbers. Subtracting $2k$ from each part, we get $n-2kr=(\lambda_1-2k,\lambda_2-2k,\ldots,\lambda_r-2k)$

Hence $n-2kr=(\lambda_1-2k,\lambda_2-2k,\ldots,\lambda_r-2k)$ is a $r-M1partition$ of $n-2kr$ with even parts not repeated and smallest parts are odd.

Therefore the number of $r-M1partitions$ of n with parts greater than or equal to $2k+1$ is $M^1p_r(n-2kr)$.

Hence $M^1p_r(2k+1,n)=M^1p_r(n-2kr)$.

2.3. Theorem: $$\sum_{n=1}^{\infty}M1spt(n)q^n=\sum_{n=1}^{\infty}\frac{q^{2n-1}\left(-q^{2n};q^2\right)_\infty}{\left(1-q^{2n-1}\right)^2\left(q^{2n+1};q^2\right)_\infty}$$

Proof: From theorem (2.1) we have

$$M1spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}M^1p\left(2k-1,n-(2k-1)t\right)+\beta$$

first replace $2k+1$ by $2k-1$, then replace n by $n-(2k-1)t$ in theorem $(2.2.)$

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}M^1p_r\left(n-(2k-1)t-r(2k-2)\right)+\beta$$

where $\beta=\begin{cases}1 & \text{if } 2k-1 \mid n\\ 0 & \text{otherwise}\end{cases}$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+(2k-1)t+r(2k-2)}\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}+\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{(2k-1)t+r(2k-1)}\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}+\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{(2k-1)t}\left[\sum_{r=1}^{\infty}\frac{\left(q^{2k-1}\right)^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}\right]+\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}}\left[\sum_{r=1}^{\infty}\frac{\left(q^{2k-1}\right)^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}\right]+\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}}\left[1+\sum_{r=1}^{\infty}\frac{\left(q^{2k-1}\right)^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}\right]$$

Put $t=q^{2k-1}, a=-q, q=q^2$ in theorem 2.1 'The Theory of partitions' by G.E.Andrews

$$=\sum_{k=1}^{\infty}\frac{q^{2k-1}}{\left(1-q^{2k-1}\right)}\prod_{r=0}^{\infty}\frac{\left(1+q^{2r+2k-1+1}\right)}{\left(1-q^{2r+2k-1}\right)}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k-1}}{\left(1-q^{2k-1}\right)}\prod_{r=0}^{\infty}\frac{\left(1+q^{2r+2k}\right)}{\left(1-q^{2r+2k-1}\right)}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k-1}\left(1+q^{2k}\right)\left(1+q^{2k+2}\right)\left(1+q^{2k+4}\right)\left(1+q^{2k+6}\right)\ldots}{\left(1-q^{2k-1}\right)\left(1-q^{2k-1}\right)\left(1-q^{2k+1}\right)\left(1-q^{2k+3}\right)\left(1-q^{2k+5}\right)\ldots}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k-1}\left(-q^{2k};q^2\right)_\infty}{\left(1-q^{2k-1}\right)^2\left(q^{2k+1};q^2\right)_\infty}$$

$$\sum_{n=1}^{\infty}M1spt(n)q^n=\sum_{n=1}^{\infty}\frac{q^{2n-1}\left(-q^{2n};q^2\right)_\infty}{\left(1-q^{2n-1}\right)^2\left(q^{2n+1};q^2\right)_\infty}$$

2.4 .Corollary: $c_1M1spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}M^1p\left(c_1(2k-1),n-c_1(2k-1)t\right)+\beta_1$

where $\beta_1=\begin{cases}1 & \text{if } c_1(2k-1)\mid n\\ 0 & \text{otherwise}\end{cases}$ and $c_1=2c-1, c\in\mathrm{N}$

2.5. Theorem: $M^1p_r\left(2c_1k+1,n\right)=M^1p_r\left(n-2c_1kr\right)$ where $c_1=2c-1, c\in\mathrm{N}$

Proof: Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r),\ \lambda_i>2c_1k\quad\forall i$, be any $r-M^1\,partition$ of n such that even numbers not repeated and smallest parts are odd numbers and c is a constant. Subtracting $2c_1k$ from each part, we get $n-2c_1kr=\left(\lambda_1-2c_1k,\lambda_2-2c_1k,\ldots,\lambda_r-2c_1k\right)$

Hence $n-2c_1kr=\left(\lambda_1-2c_1k,\lambda_2-2c_1k,\ldots,\lambda_r-2c_1k\right)$ is a $r-M^1\,partition$ of $n-2c_1kr$ with even parts not repeated and smallest parts are odd.

Therefore the number of $r-M^1\,partitions$ of n with parts greater than or equal to $2c_1k+1$ is $M^1p_r\left(n-2c_1kr\right)$.

Hence $M^1 p_r(2c_1k+1,n)=M^1 p_r(n-2c_1kr)$ where $c_1=2c-1, c\in \mathrm{N}$

2.6.Theorem:

$$\sum_{n=1}^{\infty} c_1 M1spt(n)q^n = \sum_{n=1}^{\infty} \frac{q^{c_1(2n-1)}\left(-q^{c_1(2n-1)+1};q^2\right)_\infty}{\left(1-q^{c_1(2n-1)}\right)^2\left(q^{c_1(2n-1)+2};q^2\right)_\infty} \text{ where } c_1=2c-1$$

2.7. Theorem: $\sum_{n=1}^{\infty} sumM1spt(n)q^n = \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}\left(-q^{2n};q^2\right)_\infty}{\left(1-q^{2n-1}\right)^2\left(q^{2n+1};q^2\right)_\infty}$

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