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ON ZERO-ONE INFLATED GEOMETRIC DISTRIBUTION

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ABSTRACT

In many sampling involving non negative integer data, the zeros are observed to be significantly higher than the expected assumed model. Such models are called zero-one inflated models. The zero inflated geometric distribution was recently considered and studied due to its empirical needs and application. In this paper, an extension to the case of zero inflated case is considered, namely, the zero and one inflated geometric distribution, along with some of its structural properties, and estimation of its parameters using the methods of moments and maximum likelihood estimators were obtained with three empirical examples as well.

KEYWORDS: Geometric Distribution, Inflated Model, Moments Estimator, Maximum likelihood Estimator, Inflated Geometric Distribution.

1. Introduction

The geometric distribution is a well-known discrete distribution that has been studied by many researchers due to its empirical applications. A special case of the geometric distribution arises in the researchers literature as a statistical model in application situations involving the frequency of the observed zeros, and the zeros and ones jointly, are significantly higher than the predicated frequency by the standard distribution, hence, as a result of mis-specifying the proper statistical model, leads to the so called the inflated distributions, namely the zero inflated and the zero-one inflated, respectively.

Recently, models based on zero-inflated geometric distribution were studied by many researchers. In particular, Pandey and Tiwari [1]uses an inflated geometric model consists of a mixture of a displaced geometric distribution and a logarithmic distribution model to estimate the total number of migrants in household cohort (including international migrants) of the rural areas of ComillaDistrict of Bangladesh. Saengthong et al [2] introduce the zero inflated negative binomial – crack distribution, consists of a mixture of Bernoulli distribution and negative binomial distribution, which is an alternative distribution for the excessive zero counts and overdispersion, and studied some of its properties and parameter estimates. Sharma and Landge[3]used the zero inflated negative binomial regression for modeling heavy vehicle crash rate on Indian rural highway.Pandya et al [4]proposed change point model on zero inflated geometric distribution to represent thedistribution of count data with change a point and have obtain Bays estimates of its parameters.Aryal[5]used an approximation of inflated geometric distribution to study the distribution of rural out-migrants from a household in order to help planners and policy makers for designing more effective and equitable rural and urban policies.Alshkaki[6]introduced an extension to the zero-inflated models, in which not only the number of frequencies with zeros is inflated, but the number of frequencies with ones are also inflated as well. He called such models zero-one inflated models, he studied its structure properties, as well as its relation to the standard and the zero inflated cases.See Zelterman[7], Johnson et al [8], and Forbes et al [9]for more details.

In this paper, we give in Section 2, the definition of the geometric distribution, then, in Section 3, we introduce the class of zero-one inflated geometric distribution, and some of its structural properties, namely, its mean, variance,andgenerating functions, were given in Section 4.Then in Section 5, we consider moment estimators method of itsparameters, followed by the maximum likelihood estimators method for its parameters also in Section 6.Finally, empirical examples consist of estimation of the parameters of the zero-oneinflated geometric distribution as well as fitting its frequencies were presented in Section 7, using three different sets of data representing; migrants in household cohort data, consumer credit behavior data, and heavy vehicle traffic accident data.Finally, some concluding remarks were given in Section 8.

2.GeometricDistribution

Let $\theta \in (0,1)$, then the discrete random variable (rv) X having probability mass function (pmf);

$$P(X = x) = (1 - \theta)\theta^{x} \qquad x = 0, 1, 2, \ldots \qquad (2.1)$$

is said to have a geometric distribution (GD) with parameter θ,and will denote thatby writing $X \sim GD(\theta)$.

3. Zero-One Inflated Geometric Distribution

Let X ~ GD(θ)as given in (2.1), let $\alpha \in (0,1)$ be an extra proportion added to the proportion of zero of the rv X, and letβ $\in (0,1)$ be an extra proportion added to the proportion of ones of the rv X, such that $0 < \alpha + \beta < 1$, then the rv Z having pmf defined by;

$$P(Z=z) = \begin{cases} \alpha + (1-\alpha-\beta)(1-\theta), & z = 0 \\ \beta + (1-\alpha-\beta)(1-\theta)\theta, & z = 1 \\ (1-\alpha-\beta)(1-\theta)\theta^z, & z = 2,3,4,\ldots \\ 0 & otherwise, \end{cases} \tag{3.1}$$

is said to have a zero-one inflated geometricdistribution, and will denotethat by writing Z ~ ZOIGD(θ; α, β).

Note that, if β → 0, the (3.1) reduces to the form of the zero-inflated GD. Similarly, the case with α → 0 and β → 0, reduces to the standard case of GD.

4.Some Structural Properties

Let the rv $Z \sim$ ZOIGD$(\theta;\alpha,\beta)$, then it is easy to find that;

$$E(Z) = \beta + (1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right) \tag{4.1}$$

$$= \beta + (1-\alpha-\beta)E(X)$$

$$= \beta[1-E(X)] + E(Y)$$

$$= \beta\left(\frac{1-2\theta}{1-\theta}\right) + E(Y)$$

wherethe rv X ~ GD(θ) andthe rv Y~ ZOIGD$(\theta;\alpha,0)$,that is a zero-inflated GD, and that;

$$Var(Z) = \beta(1-\beta) + (1-2\beta)(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right) + (1+\alpha+\beta)(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right)^2$$

$$= \beta(1-\beta) + (1-2\beta)(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right) + \theta[1-(\alpha+\beta)^2]Var(X)$$

$$= Var(Y) - (1-\alpha)Var(X) + \beta(1-\beta) + [(2-\beta)(1-\alpha-\beta) - \alpha\beta]\left(\frac{\theta}{1-\theta}\right)$$

$$= Var(Y) + \beta(1-\beta) + [(2-\beta)(1-\alpha-\beta) - \alpha\beta]\left(\frac{\theta}{1-\theta}\right) - (1-\alpha)\frac{\theta}{(1-\theta)^2}$$

The probability generating function $G_Z(s)$ and the moment generating function $M_Z(t)$, are respectively, given by:

$$G_Z(s) = E(s^Z)$$

$$= \alpha + \beta s + (1-\alpha-\beta)\left(\frac{1-\theta}{1-\theta s}\right) \tag{4.3}$$

and

$$M_Z(t) = E(e^{tZ})$$

$$= \alpha + \beta e^t + (1-\alpha-\beta)\left(\frac{1-\theta}{1-\theta e^t}\right)$$

5.MomentEstimators

Using the moment generating function, or obtaining them directly, the first three distribution moments about the origin for the ZOIGDcan be found to be,

$$\mu_1' = \beta + (1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right)$$

$$\mu_2' = \beta + (1-\alpha-\beta)\left[\frac{\theta(1+\theta)}{(1-\theta)^2}\right]$$

and,

$$\mu_3' = \beta + (1-\alpha-\beta)\theta\left[\frac{1+4\theta+\theta^2)}{(1-\theta)^3}\right]$$

Let $z_1, z_2, \ldots, z_n$ be a random sample from ZOIGD as given by (3.1), and let,

$$m_k' = \frac{\sum_{i=1}^{n} z_i^k}{n}, \qquad k = 1, 2, 3. \tag{5.1}$$

betheir sample moments about the origin, then solving the following simultaneous:

$$m_1' = \beta + (1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right) \tag{5.2}$$

$$m_2' = \beta + (1-\alpha-\beta)\frac{\theta(1+\theta)}{(1-\theta)^2} \tag{5.3}$$

$$m_3' = \beta + (1-\alpha-\beta)\theta\left[\frac{1+4\theta+\theta^2}{(1-\theta)^3}\right] \tag{5.4}$$

In order to solve (5.2) to (5.4) in term of θ, α, and β, let us consider the following way. It is easy to findthat the following factorial moment of the rv $Z \sim \text{ZOIGD}(\theta;\alpha,\beta)$ defined by

$$\mu_{[2]} = E[Z(Z-1)]$$

$$\mu_{[3]} = E[Z(Z-1)(Z-2)]$$

Are given by

$$\mu_{[2]} = 2(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right)^2 \quad (5.5)$$

$$\mu_{[3]} = 6(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right)^3 \quad (5.6)$$

Let fork $= 2, 3, \ldots,$

$$m'_{[k]} = \frac{\sum_{i=1}^{n} \mathrm{z}_i(\mathrm{z}_i-1)(\mathrm{z}_i-2)\ldots(\mathrm{z}_i-k+1)}{n}, \quad (5.7)$$

be k^{th}sample factorial moments, then equating the distributional factorial moments $\mu_{[2]}$and $\mu_{[3]}$ given (5.5) and (5.6), respectively, with their sample factorial moments given by (5.7),we have,

$$m'_{[2]} = 2(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right)^2 \quad (5.8)$$

$$m'_{[3]} = 6(1-\alpha-\beta)\left(\frac{\theta}{1-\theta}\right)^3$$

It follows that,

$$\frac{\mathrm{m}'_{[3]}}{\mathrm{m}'_{[2]}} = 3\left(\frac{\theta}{1-\theta}\right) \quad (5.9)$$

Hence,

$$\theta = \frac{m'_{[3]}}{m'_{[3]} + 3m'_{[2]}}$$

Or equivently,

$$\hat{\theta} = \frac{m'_3 - 3m'_2 + 2m'_1}{m'_3 - m'_1} \quad (5.10)$$

From (5.8), we have that,

$$1-\alpha-\beta = \frac{m'_{[2]}}{2}\left(\frac{1-\theta}{\theta}\right)^2 \quad (5.11)$$

And hence, it follows from (5.2) with the using of (5.11) that,

$$\hat{\beta} = m'_1 - \frac{m'_{[2]}}{2}\left(\frac{1-\theta}{\theta}\right) \quad (5.12)$$

Therefore, using (5.10), (5.12) reduces to,

$$\hat{\beta} = m'_1 - \frac{3\left(m'_2 - m'_1\right)^2}{2\left(m'_3 - 3m'_2 + 2m'_1\right)} \quad (5.13)$$

Finally, from (5.11), we have that,

$$\alpha = 1 - \beta - \frac{m'_{[2]}}{2}\left(\frac{1-\theta}{\theta}\right)^2 \tag{5.14}$$

It follows from (5.14) with the using of (5.13) and (5.10) that,

$$\hat{\alpha} = 1 - m'_1 + \frac{3(m'_3 + 5m'_1)(m'_2 - m'_1)^2}{2(m'_3 - 3m'_2 + 2m'_1)^2} \tag{5.15}$$

Hence, the moment estimates (ME) of the parameters θ, α and β are given by (5.10), (5.15) and (5.13), respectively.

6.Maximum LikelihoodEstimators

Let $z_1, z_2, \dots, z_n$ be a random sample from ZOIGD as given by (3.1), and let for i=1, 2, … n,

$$\alpha_i = \begin{cases} 1 & \text{if } z_i = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_i = \begin{cases} 1 & if\ z_i = 1, \\ 0 & otherwise \end{cases}$$

Then, for i=1, 2, … , n, (3.1) can be written, for $z_i = 0, 1, 2, \dots$, in the following form;

$$P(Z_i = z_i) = \{\alpha + (1-\alpha-\beta)(1-\theta)\}^{\alpha_i}\{\beta + (1-\alpha-\beta)(1-\theta)\theta\}^{\beta_i}$$
$$\{(1-\alpha-\beta)(1-\theta)\theta^{z_i}\}^{1-\alpha_i-\beta_i}$$

Hence, the likelihood function $L = L(\theta, \alpha, \beta; z_1, z_2, \dots, z_n)$will be,

$$L = \prod_{i=1}^{n}\{\alpha + (1-\alpha-\beta)(1-\theta)\}^{\alpha_i}\{\beta + (1-\alpha-\beta)(1-\theta)\theta\}^{\beta_i}$$
$$\{(1-\alpha-\beta)(1-\theta)\theta^{z_i}\}^{1-\alpha_i-\beta_i}$$
$$= \{\alpha + (1-\alpha-\beta)(1-\theta)\}^{n_0}\{\beta$$
$$+ (1-\alpha-\beta)(1-\theta)\theta\}^{n_1}\prod_{i=1}^{n}\{(1-\alpha-\beta)(1-\theta)\theta^{z_i}\}^{c_i}$$

where $c_i = 1 - \alpha_i - \beta_i$, $n_0 = \sum_{i=1}^{n}\alpha_i\ and n_1 = \sum_{i=1}^{n}\beta_i$. Note that n_0 and n_1 represents, respectively, the number of zeros and the number of ones in the sample. Therefore,

$$log\ L = n_0 log\{\alpha + (1-\alpha-\beta)(1-\theta)\} + n_1 log\{\beta + (1-\alpha-\beta)(1-\theta)\theta\}$$
$$+ n_c \log(1-\alpha-\beta) + n_c log(1-\theta) + \sum_{i=1}^{n} c_i\, z_i\, \log(\theta)$$

It follows that,

$$\frac{\partial}{\partial\alpha}logL = \frac{n_0\theta}{\alpha+(1-\alpha-\beta)(1-\theta)} - \frac{n_1(1-\theta)\theta}{\beta+(1-\alpha-\beta)(1-\theta)\theta} - \frac{n_c}{1-\alpha-\beta} \tag{6.1}$$

And hence,

$$\frac{\partial^2}{\partial\alpha^2}logL = -\frac{n_0\theta^2}{[\alpha+(1-\alpha-\beta)(1-\theta)]^2} - \frac{n_1[(1-\theta)\theta]^2}{[\beta+(1-\alpha-\beta)(1-\theta)\theta]^2} - \frac{n_c}{(1-\alpha-\beta)^2}$$

Therefore, $\frac{\partial^2}{\partial\alpha^2}\log L < 0$, which indicates that L has a local maximum at α. Similarly,

$$\frac{\partial}{\partial\beta}logL = -\frac{n_0(1-\theta)}{\alpha+(1-\alpha-\beta)(1-\theta)} + \frac{n_1[1-(1-\theta)\theta]}{\beta+(1-\alpha-\beta)(1-\theta)\theta} - \frac{n_c}{1-\alpha-\beta} \tag{6.2}$$

$$\frac{\partial^2}{\partial\beta^2}logL = -\frac{n_0(1-\theta)^2}{[\alpha+(1-\alpha-\beta)(1-\theta)]^2} - \frac{n_1[1-(1-\theta)\theta]^2}{[\beta+(1-\alpha-\beta)(1-\theta)\theta]^2} - \frac{n_c}{(1-\alpha-\beta)^2}$$

And hence, $\frac{\partial^2}{\partial\beta^2}\log L < 0$, which indicates that L has a local maximum at β.Finally,

$$\frac{\partial}{\partial\theta}logL = -\frac{n_0(1-\alpha-\beta)}{\alpha+(1-\alpha-\beta)(1-\theta)} + \frac{n_1(1-\alpha-\beta)(1-2\theta)}{\beta+(1-\alpha-\beta)(1-\theta)\theta} - \frac{n_c}{1-\theta} + \frac{\sum_{i=1}^{n} c_i\, z_i}{\theta} \tag{6.3}$$

with,

$$\frac{\partial^2}{\partial\theta^2}logL = -\frac{n_0(1-\alpha-\beta)^2}{[\alpha+(1-\alpha-\beta)(1-\theta)]^2} - \frac{n_1(1-\alpha-\beta)^2(1-2\theta)^2}{[\beta+(1-\alpha-\beta)(1-\theta)\theta]^2} - \frac{2n_1(1-\alpha-\beta)}{[\beta+(1-\alpha-\beta)(1-\theta)\theta]^2} - \frac{n_c}{(1-\theta)^2} - \frac{\sum_{i=1}^{n} c_i\, z_i}{\theta^2}$$

Hence, $\frac{\partial^2}{\partial\theta^2}\log L < 0$, which indicates that L has a local maximum at θ.

Letting $\frac{\partial}{\partial\alpha}\log L = 0$, we have from (6.1) that

$$1-\alpha-\beta = \frac{n_c}{\frac{n_0}{p_0}\theta - \frac{n_1}{p_1}(1-\theta)\theta} \tag{6.4}$$

where,

$$p_0 = \alpha+(1-\alpha-\beta)(1-\theta), \tag{6.5}$$

and

$$p_1 = \beta+(1-\alpha-\beta)(1-\theta)\theta \tag{6.6}$$

Setting $\frac{\partial}{\partial\theta}\log L = 0$, then (6.3) reduces, with the using of (6.5) and (6.6), to;

$$-\frac{n_0}{p_0}(1-\alpha-\beta)+\frac{n_1}{p_1}(1-\alpha-\beta)(1-2\theta)=\frac{n_c}{1-\theta}-\frac{\sum_{i=1}^{n} c_i\, z_i}{\theta} \tag{6.7}$$

Now, if we replace, p_0 and p_1 by their sample relative frequencies, i.e. by their sample estimates, the proportion of zeros and the proportion of ones in the sample, that is; $\widehat{p_0} = n_0/n$ and $\widehat{p_1} = n_1/n$, respectively, then (6.4) and (6.7) reduce respectively to;

$$1-\alpha-\beta = \frac{n_c}{n\theta^2} \tag{6.8}$$

and

$$-2n\theta(1-\alpha-\beta)=\frac{n_c}{1-\theta}-\frac{\sum_{i=1}^{n} c_i\, z_i}{\theta} \tag{6.9}$$

Hence, (6.9) with the using of (6.8) becomes;

$$-2\frac{n_c}{\theta}=\frac{n_c}{1-\theta}-\frac{\sum_{i=1}^{n} c_i\, z_i}{\theta}$$

From which we get that,

$$\hat{\theta}=\frac{\sum_{i=1}^{n} c_i\, z_i - 2n_c}{\sum_{i=1}^{n} c_i\, z_i - n_c} \tag{6.10}$$

Similarly, using (6.5), (6.6) and (6.8), the estimates of α and βare given by;

$$\hat{\alpha}=\frac{n_0}{n}-\frac{n_c}{n}\left(\frac{1-\hat{\theta}}{\hat{\theta}^2}\right) \tag{6.11}$$

and,

$$\hat{\beta}=\frac{n_1}{n}-\frac{n_c}{n}\left(\frac{1-\hat{\theta}}{\hat{\theta}}\right) \tag{6.12}$$

Hence, the maximum likelihood estimates (MLE) of the parameters θ, α and β are given by (6.10), (6.11) and (6.12), respectively.

7.Empirical Examples

In this Section, three different sets of data will be used to estimate empirically the parameters of the ZOIGD as well as fitting its frequencies.

7.1 Migrants in Household Cohort Data

Table (1) shows the number of migrants in household cohort (including International migrants) of the Rural Areas of Comilla district of Bangladesh, that has used by Pandey and Tiwari [1]to estimate the ML and the MLE estimates of the parameters of their proposed model (PTM) consisting of a mixture of a displaced geometric distribution and a logarithmic distribution model, in order to estimate the frequencies of migrants per household including the international migrants, as well as the ZOIG model estimates.

Table (1) *The ML and the MLE estimates of the PTM* and the ZOIGDfor the number of migrants in household cohort (including International migrants) of the Rural Areas of Comilla district of Bangladesh.*

No. of Migrants per Household	Observed Frequencies	Expected Frequencies			
		ME		MLE	
		PTM	ZOIGD	PTM	ZOIGD
0	1941	1941	1942	1941	1941
1	544	544	551	544	544
2	117	112	100	112	108
3	50	49	51	49	53
4	18	25	26	25	26
5+	26	25	26	25	23
Total	2696	2696	2696	2696	2696
Model Parameters	θ	---	0.515695	---	0.532895
	α	---	0.70486	---	0718917
	β	---	0.0545025	---	0.0705393
	χ^2	1.052057	2.870848	1.052057	0.5504899
	df	5	5	5	5
	p-value	0.958258	0.719889	0.958258	0.990158

The p-values forχ^2goodness of fitting a ZOIGD model fitting and Pandey and Tiwari [1]model's fitting, as shown in Table (1), indicate that the ZOIGD model give accurate fitting as Pandey and Tiwari [1]model does.

7.2Consumer Credit Behavior Data

Table (2) shows the MLE of the parameters of Saengthong et al [21]ZINB and ZINB-CR models as well as of the ZOIGD model, to estimate the frequencies of number of major derogatory reports in the credit history of individual credit card applicants, from which we can see the accuracy of the ZOIGD model.

Table (2) *The MLE estimates of the ZINB, ZINB-CR and the ZOIGDfor the number of major derogatory reports in the credit history of individual credit card applicants.*

No. of Major Derogatory Reports	Observed Frequencies	Expected Frequencies		
		MLE		
		ZINB	ZINB-CR	ZOIGD
0	1060	1060	1062	1060
1	137	94	135	137
2	50	73	55	47
3	24	45	27	29
4	17	25	15	18
5	11	12	9	11
6+	20	10	16	17
Total	1319	1319	1321	1319
Model Parameters	θ	---	---	0.615089
	α	---	---	0.709537
	β	---	---	0.045985
	χ^2	49.36012	2.532386	1.638526
	df	6	6	6
	p-value	0.00000	0.864825	0.949767

7.3Heavy Vehicle Traffic Accident Data

Sharma and Landge[3]used zero inflated negative binomial regression as their model to estimate the accident frequencies for the heavy vehicle traffic accident data collected for the year 2010, as given in their Table 5. Using the ZOIG model to estimate the accident frequencies, as given in

Table (3), shows accurate estimates for the given accident data using the MLE.

Table (3) *The parameters estimates of the ZINB RegressionofSharma and Landge (2013)and the ZOIGfor Heavy Vehicle Traffic Accident Data.*

No. of Accident	Observed Frequencies	Expected Frequencies	
		ZINB Regression Model	MLE ZOIG
0	55	64	55
1	26	17	26
2	4	11	6
3	3	7	4
4	3	4	2
5	1	2	1
6	3	1	1
7+	1	1	1
Total	96	107	96
Model Parameters	θ	---	0.615383
	α	---	0.414225
	β	---	0.173177
	χ^2	17.52059	2.75
	df	7	7
	p-value	0.014333	0.6093

8. Conclusions

We considered estimation of the parameters of the zero-one inflated geometric distributions by the method of moment estimators and maximum likelihood estimators. The method of maximum likelihood estimators was shown to have butter estimates on three different real data sets, namely, the number of migrants in household cohort, consumer credit behavior data, and a heavy vehicle traffic accident data. The zero-one inflated geometric distribution is shown also to have a better fitting for that frequencies of the real data sets than the zero inflated geometric distribution.

References

1. R. S. A. Alshkaki,An Extension to the Zero-Inflated Generalized Power SeriesDistributions. To be presented at the International Conference on Statistical Distributions and Applications 2016, October 14-16,Niagara Falls, Canada.
2. T. R. Aryal,Inflated Geometric Distribution to Study the Distribution of Rural Out-Migrants. Journal of the Institute of Engineering, 8, 2011, 6–268.
3. C. Forbes, M. Evans, N. Hastings, and B. Peacock,*Statistical Distributions*. Fourth Edition (John Wiley and Sons; New Jersey, 2011).
4. N. L. Johnson, A. W. Kemp, and S.Kotz,*Univariate Discrete Distributions*, Third Edition (John Wiley and Sons; New Jersey, 2005).
5. M. Pandya, H. Pandya and S. Pandya, Bayesian Inference On Mixture Of Geometric With Degenerate Distribution: Zero Inflated Geometric Distribution. International Journal of Recent Research and Applied Studies (IJRRAS), 13(1), 2012, 53-66.
6. H. Pandey and R. Tiwari, An Inflated Probability Model for the Rural Out-Migration. Recent Research in Science and Technology, 3(7), 2011, 100-103.
7. P.Saengthong, W.Bodhisuwan, and A. Thongteeraparp, The Zero Inflated Negative Binomial – Crack Distribution: Some Properties And Parameter Estimation. Songklanakarin J. Sci. Technol, 37 (6), 2015, 701-711.
8. A.K.Sharma and V.S.Landge, Zero Inflated Negative Binomial For Modeling Heavy Vehicle Crash Rate On Indian Rural Highway. International Journal of Advances in Engineering & Technology, 5(2), 2013, 292-301.
9. D. Zelterman,*Discrete Distributions, Applications in the Health Sciences*. (John Wiley and Sons; New Jersey, 2004).