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Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

GOLDEN PROPORTIONS FOR THE GENERALIZED TETRANACCI NUMBERS

Gautam S. Hathiwala[1], Devbhadra V. Shah[2]

[1]Lecturer, Department of Applied Science, C.K. Pithawala College of Engineering and Technology, India
[2]Associate Professor, Department of Mathematics, Sir P.T. Sarvajanik College of Science, India

ABSTRACT

Tetranacci sequence is defined as $T_0 = T_1 = T_2 = 0, T_3 = 1$ *and the recurrence relation* $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$; $n \geq 4$. *Here we consider the sequence of whole family of generalized tetranacci numbers defined by recurrence relation* $T_n + T_{n+a} + T_{n+b} + T_{n+c} = T_{n+d}$; *where* $1 \leq a < b < c < d$ *are integers. Here we obtain the generalized golden proportions for the whole family of generalized tetranacci numbers. In fact we prove that* $\lim_{n \to \infty} \frac{T_{n+a+t}}{T_{n+a}} = \omega^t$; *where* t *is odd and* ω *is some real number between 1 and 2.*

KEYWORDS: TETRANACCI SEQUENCE, FIBONACCI SEQUENCE, GOLDEN PROPORTION, TRIBONACCI SEQUENCE.

1. Introduction:

It is a well-known fact that the ratio of consecutive Fibonacci numbers converges to a fixed ratio $\phi = \frac{1+\sqrt{5}}{2} = 1.61803$, the golden proportion which is the positive root of the equation $x^2 - x - 1 = 0$. Stakhov [1] defined the p - Fibonacci numbers, $F_p(n)$, by the recurrence relation

$$F_p(n) = \begin{cases} 1; & 1 \leq n \leq p+1 \\ F_p(n-1) + F_p(n-p-1); & n > p+1 \end{cases}, \text{ where } p = 1,2,3, \ldots .$$

It can be seen clearly that for $p = 1$, we get the usual Fibonacci sequence

$$F(n) = F(n-1) + F(n-2); \text{ where } F(1) = F(2) = 1.$$

The values of $F_p(n)$ for $p = 1,2,3,4,5,6$ and for first 15 values of *n* are shown in table 1.

Table 1: Values of $F_p(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_1(n)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_2(n)$	1	1	1	2	3	4	6	9	13	19	28	41	60	88	129
$F_3(n)$	1	1	1	1	2	3	4	5	7	10	14	19	26	36	50
$F_4(n)$	1	1	1	1	1	2	3	4	5	6	8	11	15	20	26
$F_5(n)$	1	1	2	1	1	1	2	3	4	5	6	7	9	12	16
$F_6(n)$	1	1	1	1	1	1	1	2	3	4	5	6	7	8	10

Stakhov [1] also proved that $F_p(n)$ satisfies

$$\lim_{n \to \infty} \frac{F_p(n)}{F_p(n-1)} = \varphi_p, \tag{1}$$

where the golden p – proportion φ_p is the root of $x^{p+1} = x^p + 1$.

De Villiers [2] made the similar observations and gave the partial proof of (1) in the case when *p* is odd, with the suggestions for the case when *p* is even. Later, Falcon [3] generalized the same problem and gave the complete proof of the same.

Also, Shah, Mehta [4] considered the similar problem for the sequence of tribonacci numbers defined by the recurrence relation $T_n + T_{n+1} + T_{n+2} = T_{n+3}$; where $n \geq 1$. They defined sequence of generalized Tribonacci numbers by the recursive relation $T_n + T_{n+p} + T_{n+q} = T_{n+r}$, where $1 \leq p < q < r$ are integers. In fact, they proved that

$$\lim_{n \to \infty} \frac{T_{n+p+k}}{T_{n+p}} = M^k \tag{2}$$

except when *k* is odd and p, q, r are any positive integers.

Lot of research has been done ([5], [6], [7], [8]) and still being pursued on the sequence of tribonacci numbers. The first 10 values of T_n for different values of p, q and r, where $1 \leq p < q < r$ are given below.

Table 2: The first 10 values of T_n for different values of p, q and r

p	q	r	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
1	2	3	0	1	1	2	4	7	13	24	44	81
1	2	4	0	1	1	2	2	4	5	8	11	17
1	2	5	0	1	1	2	4	2	4	7	8	10
1	3	4	0	1	1	2	3	5	8	13	21	34
1	3	5	0	1	1	2	4	3	6	9	12	16
2	3	4	0	1	1	2	3	6	10	18	31	55
2	3	5	0	1	1	2	4	3	7	8	12	19
3	4	5	0	1	1	2	4	6	11	18	31	53

In this paper we continue this process of generalization and define a generalized recursive formula for the sequence of Tetranacci numbers and obtain the generalized golden proportions for the same.

2. Preliminaries:

The sequence of tetranacci numbers is defined by the recurrence relation $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$; where n ≥ 4 and $T_0 = T_1 = T_2 = 0, T_3 = 1$. (3)

It is seen that the ratio of consecutive terms of tetranacci sequence converges to fixed real number. In fact, we have

$$\lim_{n \to \infty} \frac{T_{n+1}}{T_n} = 1.92756 \tag{4}$$

We now define the sequence of *generalized Tetranacci numbers* by the recurrence relation

$$T_n + T_{n+a} + T_{n+b} + T_{n+c} = T_{n+d} \tag{5}$$

where $1 \leq a < b < c < d$ are integers.

This recurrence relation gives the whole family of tetranacci sequence. Below in the table we give few tetranacci sequences for some values of a, b, c and d.

Table 3: Tetranacci numbers for some values of a, b, c and d

a	b	c	d	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
1	2	3	4	0	0	1	1	2	4	8	15	29	56
1	3	5	6	0	0	1	1	2	4	5	8	14	22
1	2	4	5	0	0	1	1	2	3	5	9	15	25
1	2	4	6	0	0	1	1	2	4	3	6	7	13
1	3	5	7	0	0	1	1	2	4	5	8	14	22
2	3	4	6	0	0	1	1	2	4	8	7	15	20
2	4	6	8	0	0	1	1	2	4	8	15	11	20
2	4	5	7	0	0	1	1	2	4	8	7	13	18
2	3	5	6	0	0	1	1	2	4	6	9	16	27
3	5	6	7	0	0	1	1	2	4	8	13	23	41
3	5	6	8	0	0	1	1	2	4	8	15	13	25
2	3	6	7	0	0	1	1	2	4	8	10	13	20
3	4	5	7	0	0	1	1	2	4	8	7	14	20

2	3	5	7	0	0	1	1	2	4	8	6	11	13
2	4	5	6	0	0	1	1	2	4	7	12	22	39
2	4	6	7	0	0	1	1	2	4	8	11	16	27

Clearly for $a = 1, b = 2, c = 3$ and $d = 4$, we get the sequence of classical tetranacci numbers. We first assume that $b = a + x$, $c = a + y$ and $d = a + z$, for some positive integers x, y & z. This clearly gives $x < y < z$. Thus result (5) can be written as

$$T_n + T_{n+a} + T_{n+a+x} + T_{n+a+y} = T_{n+a+z} .$$

Replacing n by $n + a + z$, we get

$$T_n = T_{n-a-z} + T_{n-z} + T_{n+x-z} + T_{n+y-z}.$$

For the above difference equation, the corresponding characteristic equation can be given as $\lambda^n = \lambda^{n-a-z} + \lambda^{n-z} + \lambda^{n+x-z} + \lambda^{n+y-z}$, which on simplification becomes

$$\lambda^{a+z} = \lambda^{a+y} + \lambda^{a+x} + \lambda^a + 1. \qquad (6)$$

We now write equation (6) as $\lambda^a = \frac{1}{\lambda^z - \lambda^y - \lambda^x - 1}$ with $x < y < z$.

Thus solving (6) is equivalent to solving the system

$$F(\lambda) = \frac{1}{\lambda^z - \lambda^y - \lambda^x - 1} \text{ and } G(\lambda) = \lambda^a. \qquad (7)$$

We find the intersection of the curves defined in (7) for different values of x, y and z.

3. Analysis of roots:

We now consider different cases when a, x, y and z are even or odd.

When a is even, the curve $G(\lambda) = \lambda^a$ is symmetric about y - axis. Also, it can be observed that when z is even, system (7) has two real roots, say ω, ω' such that $1 < \omega < 2$ and $-2 < \omega' < 0$ and when z is odd, system (7) has only one real root ω, where $1 < \omega < 2$.

Also if a is odd, the curve $G(\lambda) = \lambda^a$ is symmetric about origin. Also, it is been observed that when z is odd, system (7) has two real roots ω, ω' such that $1 < \omega < 2$ and $-2 < \omega' < 0$. Moreover, when z is even and when x and y are either even or odd, then system (7) have only one real root ω such that $1 < \omega < 2$. And when x, y and z are even, then system (7) has three real roots ω, ω' and ω'', such that $1 < \omega < 2$ and $-2 < \omega', \omega'' < 0$.

Following are the graphs of system (7) for all possible permutations of a, x, y and z as even and odd.

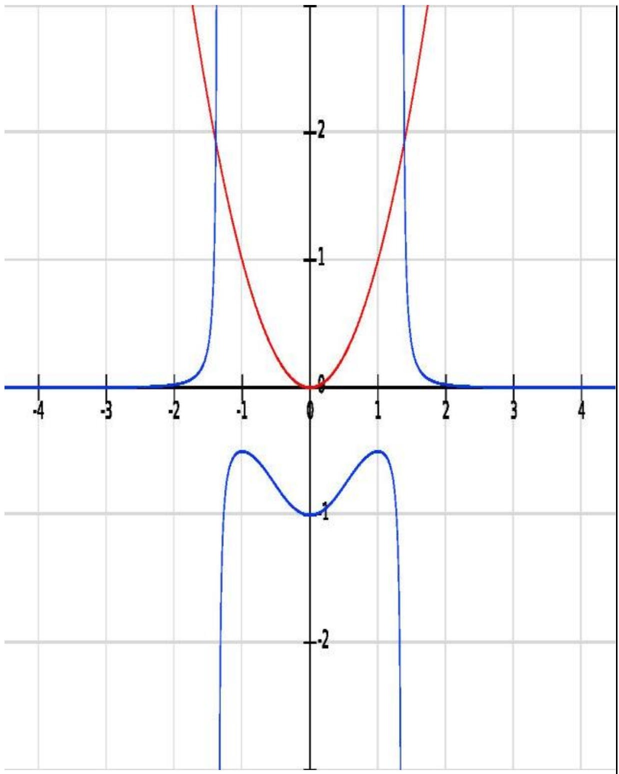

Figure 1: a, x, y, z are even

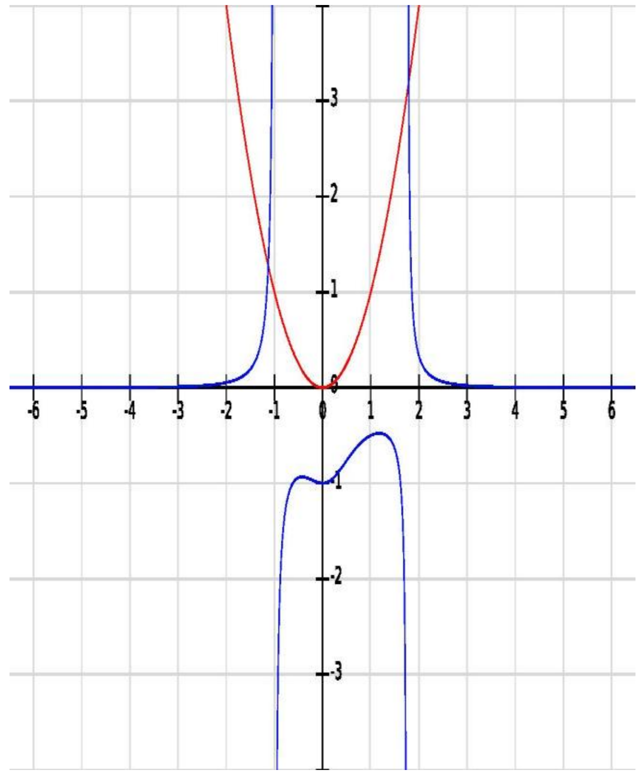

Figure 2: a, x, z are even and y is odd

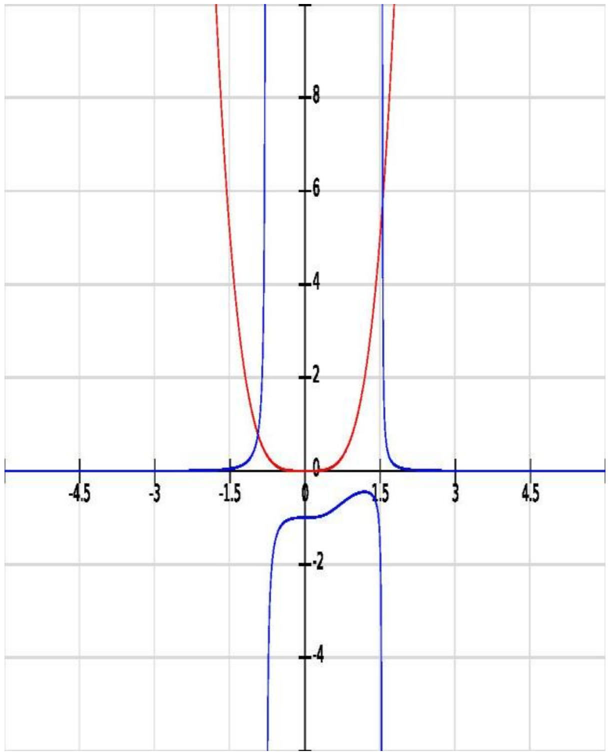

Figure 3: a, z are even and x, y are odd

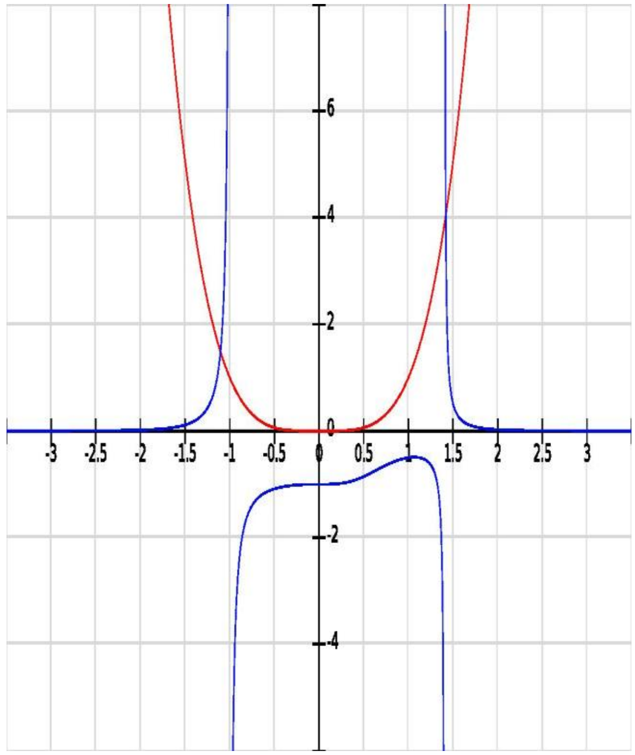

Figure 4: a, y, z are even and x is odd

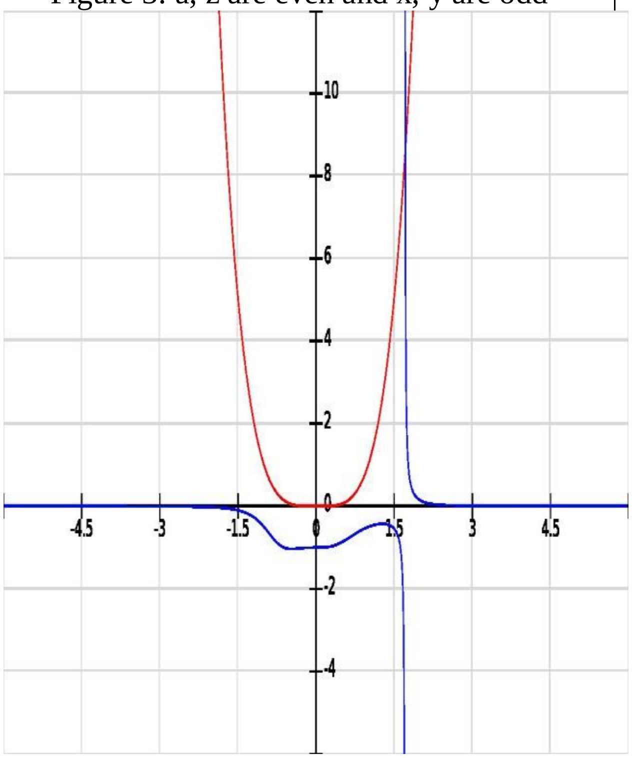

Figure 5: a, y are even and x, z are odd

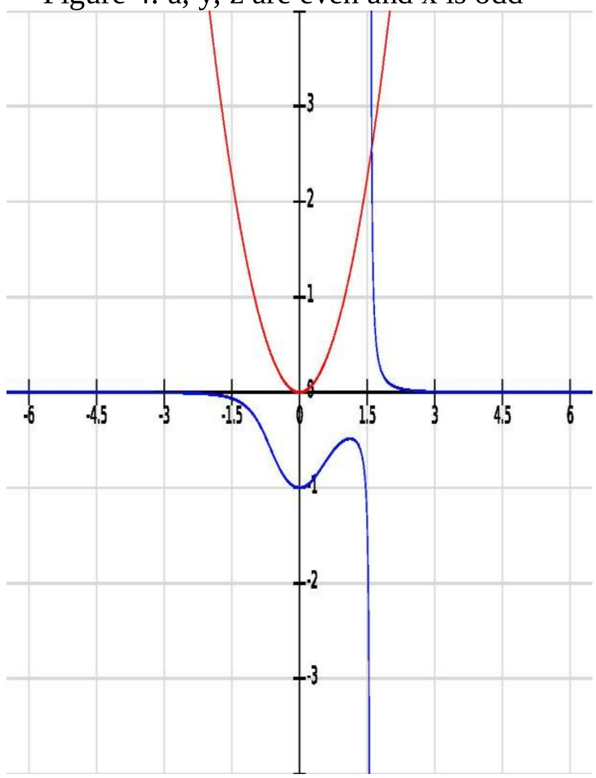

Figure 6: a, x, y are even and z is odd

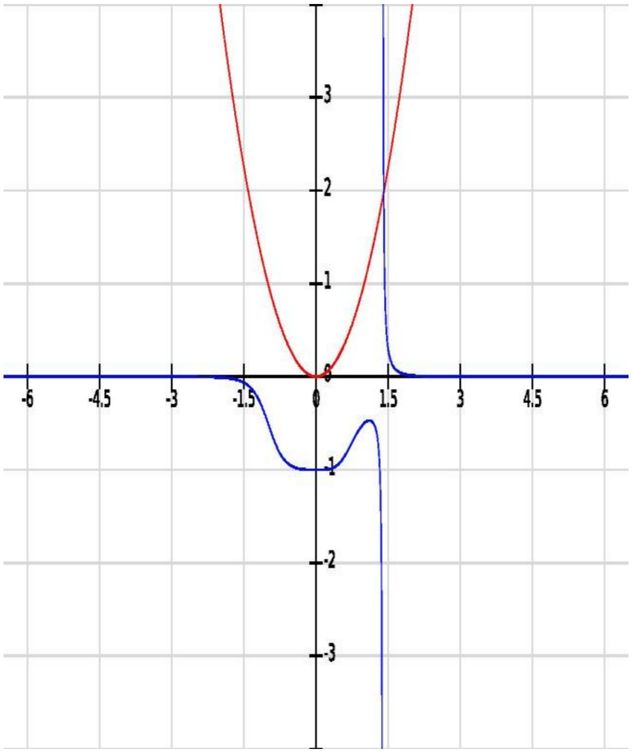

Figure 7:a, x are even and y, z are odd

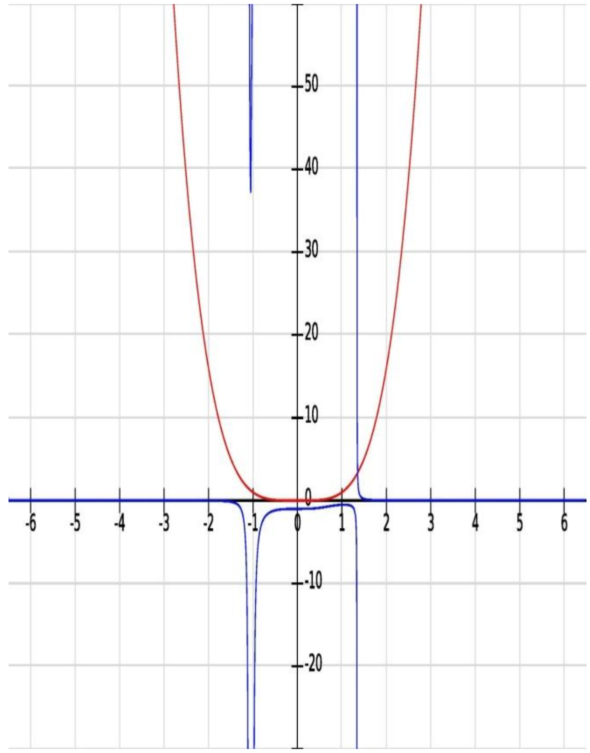

Figure 8: a is even and x, y, z are odd

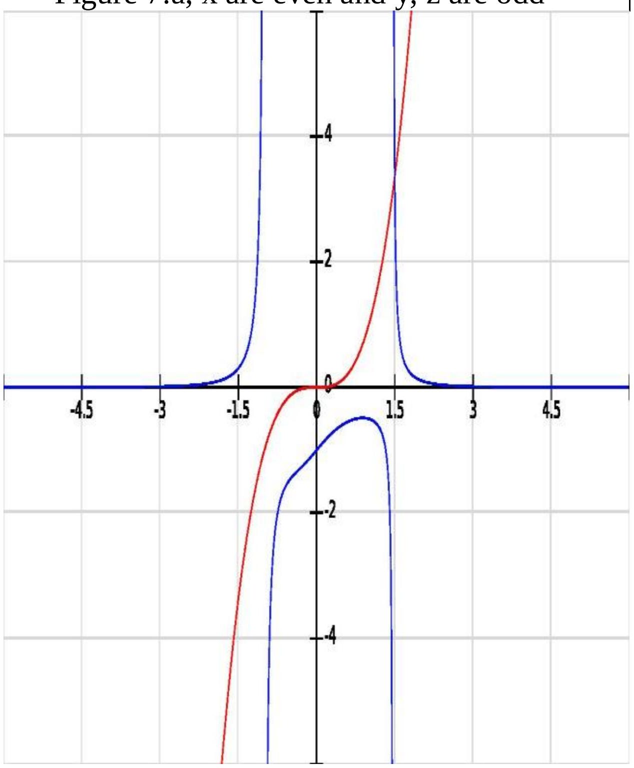

Figure 9: a is odd and x, y, z are even

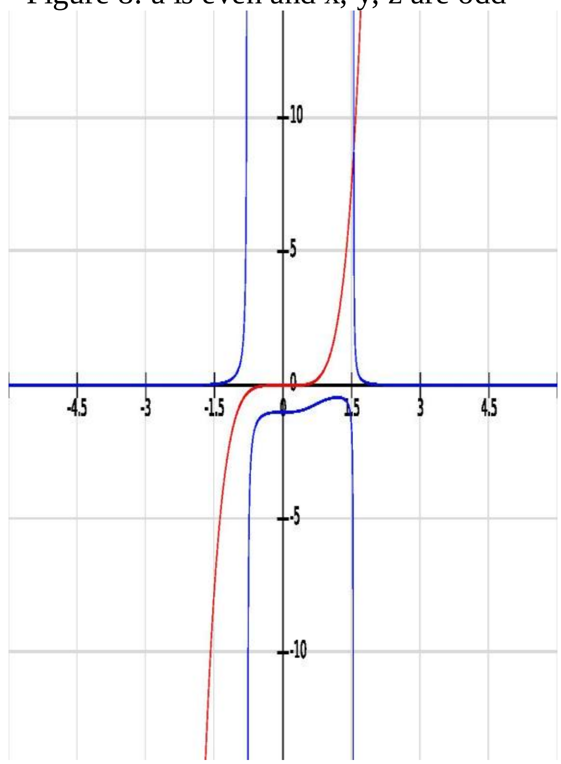

Figure 10: a, x, y are odd and z is even

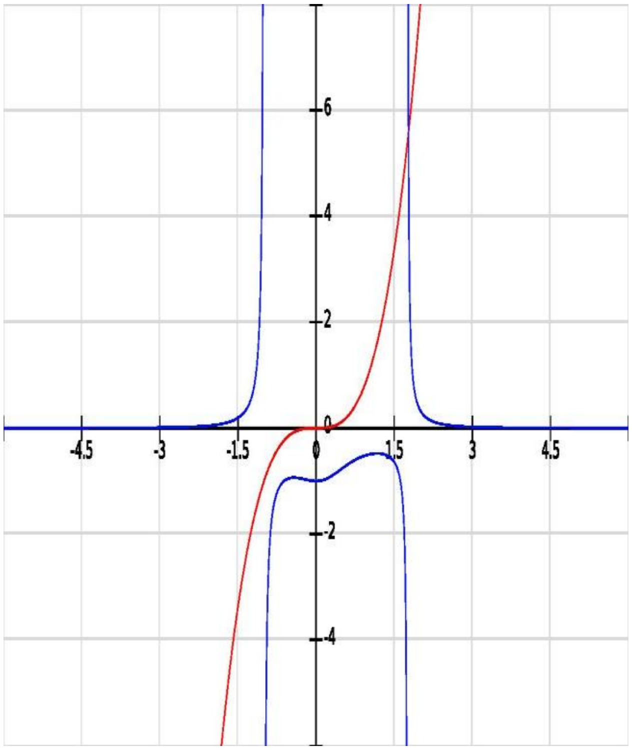

Figure 11: a, y are odd and x, z are even

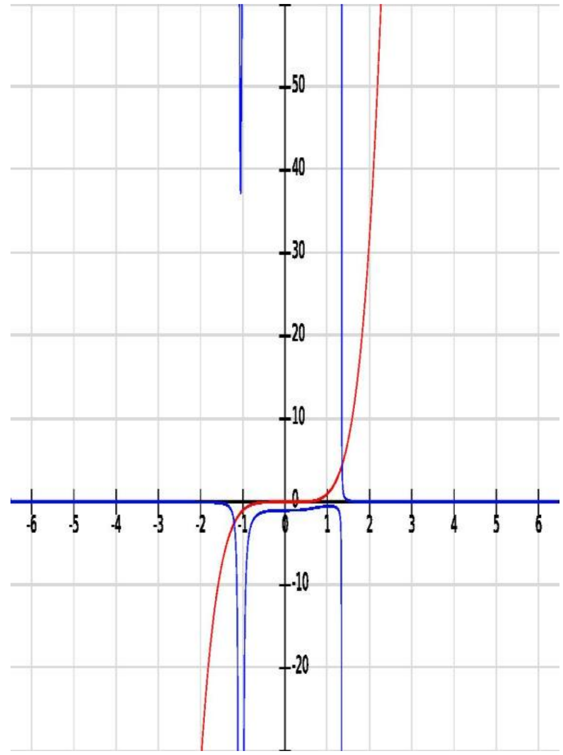

Figure 12: a, x, y, z are odd

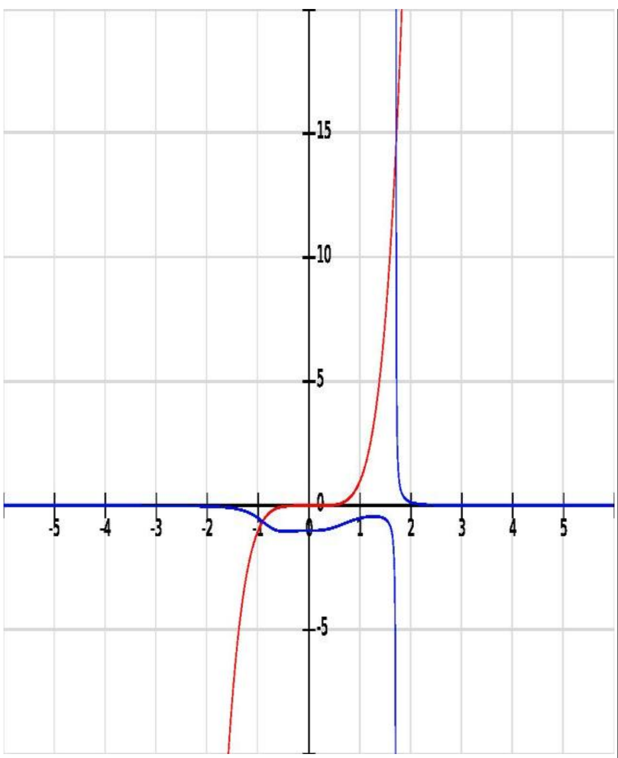

Figure 13: a, x, z are odd and y is even

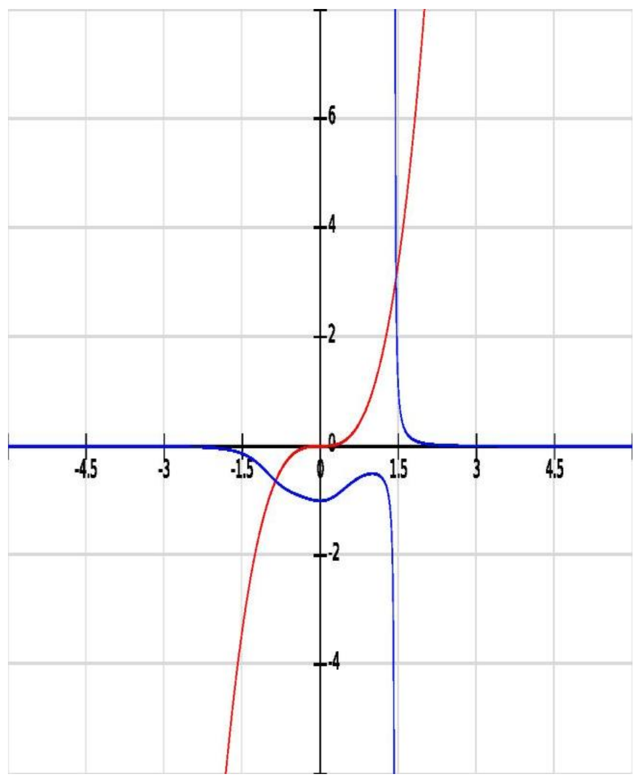

Figure 14: a, y, z are odd and x is even

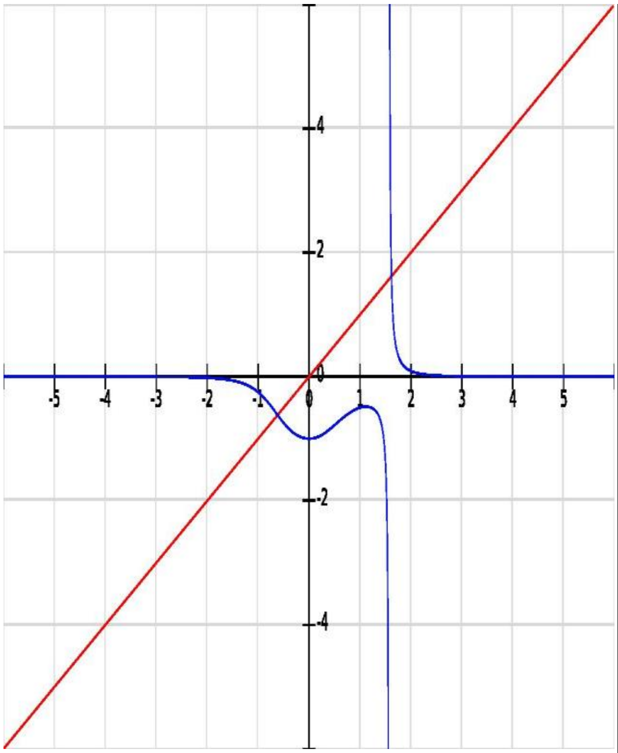

Figure 15: a, z are odd and x, y are even

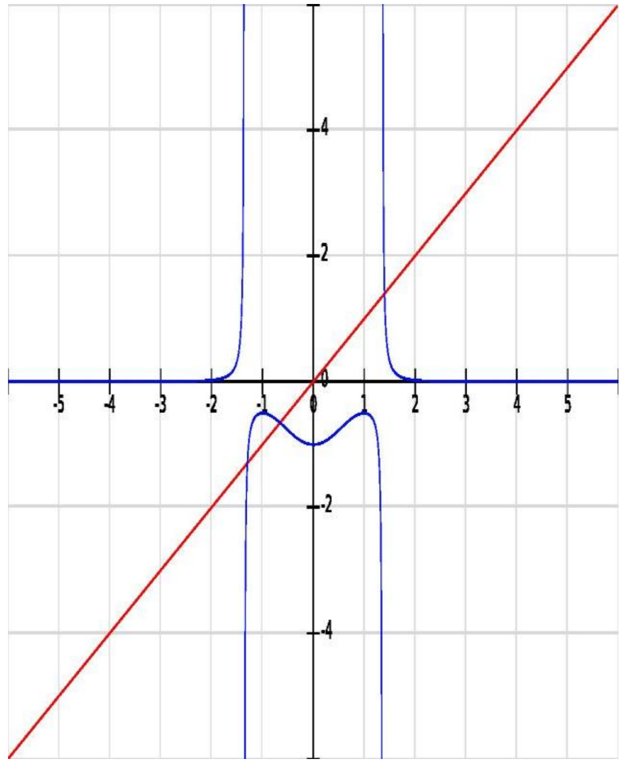

Figure 16: a is odd and x, y, z are even

From the above figures, following conclusions can be made for the real roots of system (7):

(1) One positive real root ω, such that $1 < \omega < 2$, where a is even and z is odd and x and y are either even or odd.

(2) One positive real root ω, such that $1 < \omega < 2$, where a is odd, z is even and either of x and y is even or odd otherwise both are odd.

(3) Two real roots ω and ω' such that $1 < \omega < 2$ and $-2 < \omega' < 0$, where a and z are odd and x and y are even or odd.

(4) Two real roots ω and ω' such that $1 < \omega < 2$ and $-2 < \omega' < 0$, where a and z are even and x and y are even or odd.

(5) Three real roots $\omega, \omega', \omega''$ such that $1 < \omega < 2$ and $-2 < \omega', \omega'' < 0$, where a is odd and x, y, z are even.

It should be noted that for the real roots ω, ω' and ω'' of system (7) if they exist, then we always have $|\omega| > |\omega'|$ and $|\omega| > |\omega''|$ where $1 < \omega < 2$ and $-2 < \omega', \omega'' < 0$.

We summarize the above given information in the below table:

Table: 4

a	x	y	z	No. of Real Roots	Real Roots
Even	Even	Even	Even	2	$\lambda = \pm\omega; 1 < \omega < 2$
Even	Even	Even	Odd	1	$\lambda = \omega; 1 < \omega < 2$
Even	Even	Odd	Even	2	$\lambda = \omega, \omega'; 1 < \omega < 2, \quad -2 < \omega' < 0$
Even	Even	Odd	Odd	1	$\lambda = \omega; 1 < \omega < 2$
Even	Odd	Even	Even	2	$\lambda = \omega, \omega'; 1 < \omega < 2, -2 < \omega' < 0$
Even	Odd	Even	Odd	1	$\lambda = \omega; 1 < \omega < 2$
Even	Odd	Odd	Even	2	$\lambda = \omega, \omega'; 1 < \omega < 2, \quad -2 < \omega' < 0$
Even	Odd	Odd	Odd	1	$\lambda = \omega; 1 < \omega < 2$
Odd	Even	Even	Even	3	$\lambda = \omega, \omega', \omega'', \quad 1 < \omega < 2, -2 < \omega', \omega''$
Odd	Even	Even	Odd	2	$\lambda = \omega, \omega'; 1 < \omega < 2, -2 < \omega' < 0$
Odd	Even	Odd	Even	1	$\lambda = \omega; 1 < \omega < 2$
Odd	Even	Odd	Odd	2	$\lambda = \omega, \omega'; 1 < \omega < 2, -2 < \omega' < 0$
Odd	Odd	Even	Even	1	$\lambda = \omega; 1 < \omega < 2$
Odd	Odd	Even	Odd	2	$\lambda = \omega, \omega'; 1 < \omega < 2, -2 < \omega' < 0$
Odd	Odd	Odd	Even	1	$\lambda = \omega; 1 < \omega < 2$
Odd	Odd	Odd	Odd	2	$\lambda = \omega, \omega'; 1 < \omega < 2, -2 < \omega' < 0$

It can be clearly observed that (6) as equivalently (7) has $(a + z)$ roots. The roots of (7) other than mentioned above are simple complex numbers whose modulus value is always less than ω. We express these complex roots in exponential form as $Z_j = r_j e^{i\theta_j}$; where $r_j = \sqrt{{a_j}^2 + {b_j}^2}$, where $\theta_j = \tan^{-1}\left(\frac{b_j}{a_j}\right)$. Also $r_j < 1$ for all j.

4. Few important results:

Here we prove some intermediate results which together will lead to the main result of this paper. Throughout we consider t as a fixed positive integer, x, y and zare some positive integers such that $x < y < z$ and $1 < \omega < 2$, $-2 < \omega', \omega'' < 0$, where ω, ω' and ω'' are the real roots of system (9). We note that $|\omega| > |\omega'|$ and $|\omega| > |\omega''|$.

Lemma 1: When all a, x, y and z are even then

$$\lim_{n \to \infty} \frac{T_{n+a+t}}{T_{n+a}} = \omega^t;$$

if t is even and this limit does not exist if t is odd.

Proof: Here a, x, y and z are even. Then it can be seen that the characteristic equation (6) has two real roots $\pm\omega$; where $1 < \omega < 2$.Thus, by the theory of equations [9], we write the solution of (6) as $T_n = c_1\omega^n + c_2(-\omega)^n + \sum_{j=3}^{a+z} c_j Z_j^n$; where c_j's are constants.

Then, $\lim_{n\to\infty}\frac{T_{n+a+t}}{T_{n+a}}=\lim_{n\to\infty}\frac{T_n}{T_{n-t}}=\lim_{n\to\infty}\frac{c_1\omega^n+c_2(-\omega)^n+\sum_{j=3}^{a+z}c_jZ_j^n}{c_1\omega^{n-t}+c_2(-\omega)^{n-t}+\sum_{j=3}^{a+z}c_jZ_j^{n-t}}$

$$=\lim_{n\to\infty}\frac{c_1+c_2(-1)^n+\sum_{j=3}^{a+z}c_j\left(\frac{Z_j}{\omega}\right)^n}{c_1\left(\frac{1}{\omega}\right)^t+c_2(-1)^{n-t}\left(\frac{1}{\omega}\right)^t+\sum_{j=3}^{a+z}c_j\left(\frac{Z_j}{\omega}\right)^{n-t}\left(\frac{1}{\omega^t}\right)}$$

$$=\lim_{n\to\infty}\frac{c_1+c_2(-1)^n+\sum_{j=3}^{a+z}c_j\left(\frac{r_j}{\omega}\right)^n e_j^{in\theta_j}}{\frac{1}{\omega^t}\{c_1+(-1)^{n-t}c_2\}+\sum_{j=3}^{a+z}c_j\left(\frac{r_j}{\omega}\right)^{n-t}\left(\frac{1}{\omega^t}\right)e^{i(n-t)\theta_j}}$$

Now since $r_j<1$and $1<\omega<2$, as $n\to\infty,\left(\frac{r_j}{\omega}\right)^n\to 0$ and $\left(\frac{r_j}{\omega}\right)^{n-t}\to 0$. Therefore, we get

$$\lim_{n\to\infty}\frac{T_{n+a+t}}{T_{n+a}}=\frac{c_1+(-1)^nc_2}{\frac{1}{\omega^t}\{c_1+(-1)^{n-t}c_2\}}.$$

Now when t is even, $\lim_{n\to\infty}\frac{T_{n+a+t}}{T_{n+a}}=\lim_{n\to\infty}\frac{c_1+(-1)^nc_2}{\frac{1}{\omega^t}\{c_1+(-1)^nc_2\}}=\omega^t$.

Also, when k is odd, $\lim_{n\to\infty}\frac{T_{n+a+t}}{T_{n+a}}=\omega^t\lim_{n\to\infty}\frac{c_1+(-1)^nc_2}{\{c_1-(-1)^nc_2\}}$.

Clearly, the limit does not exist when t is odd.This proves the required result.

Lemma 2: When (i) a and z are odd or (ii) a and z are even, and both of x, y are even or odd together, we have $\lim_{n\to\infty}\frac{T_{n+a+t}}{T_{n+a}}=\omega^t; t=1,2,3,\ldots$.

Proof: Here, in both the cases equation (6) has two real roots and the remaining $(a+z-2)$ roots are complex numbers. Hence, if ω and ω' are two real roots of equation (8) then by the theory of equations [9], the solution of (6) is given as $T_n=c_1\omega^n+c_2(\omega')^n+\sum_{j=3}^{a+z}c_jZ_j^n$; where c_j's are arbitrary constants.

Since $|\omega'|<|\omega|$ and $r_j<1<\omega$, we have as $n\to\infty,\left(\frac{\omega'}{\omega}\right)^n\to 0$ and $\left(\frac{r_j}{\omega}\right)^n\to 0$ and thus proceeding as above Lemma 1, we get our required result.

Lemma 3: When (i) a is even and z is odd or (ii) a is odd and z is even, and both of x,y are even or odd together, we have $\lim_{n\to\infty}\frac{T_{n+a+t}}{T_{n+a}}=\omega^t; t=1,2,3,\ldots$.

Proof: Here, since we have $a+z$ odd, equation (6) has one real roots and remaining $(a+z-1)$ roots are complex numbers. Assuming the real root of (6) as ω, we can write by the theory of equations [9], the solution of (6) as $T_n=c_1\omega^n+\sum_{j=2}^{a+k}c_jZ_j^n$; where c_j's are arbitrary constants. Proceeding as above, the result follows.

Lemma 4: When a is odd and x, y and z are even, then $\lim_{n \to \infty} \frac{T_{n+a+t}}{T_{n+a}} = \omega^t; t = 1, 2, 3, \dots$.

Proof: Here, in this case the total number of roots of (6) is $(a + z)$ of which three are real roots and remaining $(a + z - 3)$ are complex numbers. Suppose the three real roots are ω, ω' and ω'', where $1 < \omega < 2$ and $-2 < \omega', \omega'' < 0$. Therefore, by theory of equations [9], we can write $T_n = c_1\omega^n + c_2(\omega')^n + c_3(\omega'')^n + \sum_{j=4}^{a+z} c_j Z_j^n$ where c_j's are arbitrary constants. Proceeding as earlier, we get our required result.

5. Main result:

The following main result easily follows by taking into consideration all the earlier results:

Theorem 5: If a, x, y, z are any positive integers where $x < y < z$, then for the sequence of generalized tetranacci numbers, it is always true that $\lim_{n \to \infty} \frac{T_{n+a+t}}{T_{n+a}} = \omega^t$; unless t is odd.

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