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FURTHER RESULTS ON PSEUDO-D-ACHROMATIC NUMBER OF GRAPHS

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ABSTRACT

By extending the notion of pseudo-achromatic number in the context of (k,d) coloring and introducing the concept of pseudo-d-achromatic number $\psi_s^d(G)$ of a graph G. In this paper, I discuss further results on this new coloring in terms of partition graphs, the effect on the pseudo-d-achromatic number of removing points or lines, the exact values for the cycles, complete m-partite graphs and the bounds for the pseudo-d-achromatic numbers in terms of other parameters and the construction of k -edge d-critical graphs.

Keywords: Pseudo d- achromatic number, k −pseudo complete d-colorable, partition graphs, k −edge d- critical graphs.

2000 Mathematical classification: 05c15, 05c75.

1.Introduction:

Let G be a simple graph. A coloring $\zeta: V(G) \to \{1,2,3,\dots,k\}$ is called pseudo complete if each pair of different colors appears in an edge. The pseudo achromatic number $\psi_s^d(G)$ is the maximum k for which there exists a pseudo complete coloring of G. If the coloring is required to be proper(that is, each chromatic class is independent) then such a maximum is known as the achromatic number of G denoted by ψ(G). The chromatic number $\chi(G)$ is the minimum number of colors required for the vertex coloring of G. A chromatic coloring that used $\chi(G)$-colors is a complete coloring. Hence, $\chi(G) \le \psi(G) \le \psi_s(G)$ – [1].

Vince [6] introduced the concept of starchromatic number which is the natural generalization of chromatic number. Let k, d be positive integers with $k \ge 2d$. Let $z_k = \{1,2,3,\ldots,k\}$ is the set of integer modulo k and $D_k(x,y) = \min\{|x-y|, k-|x-y|\}$. A (k,d)-coloring of a graph G is a mapping $C: V \to z_k$ such that $D_k(C(u), C(v)) \ge d$ for each edge $uv \in E$. If $|V(G)| = n$ and G has a (k,d) –coloring, then the star chromatic number $\chi^*(G)$ of a graph G is defined by $\chi^*(G) = \min\{\frac{k}{d} / G$ has a $(k.d)$ –coloring and $2d \le k \le n\}$. The concept of d-achromatic number $\psi^d(G)$ and pseudo d-achromatic number $\psi_s^d(G)$ was introduced in [4] in the context of (k,d) –coloring of G. A pseudo complete d-coloring of using k colors is a mapping $\varphi: V(G) \to Z_k$ such that for any two elements $i, j \in Z_k$ with $D_k(i,j) \ge d$ there exists adjacent vertices u, v such that $\varphi(u) = i$ and $\varphi(v) = j$. The pseudo achromatic number $\psi_s^d(G)$ is the maximum value of k for which there exists a pseudo complete d-coloring of G. A graph having a pseudo complete d-coloring using k colors is called a $k-$ pseudo complete d-colorable graph. If the complete d-coloring required to be proper, then such a maximum is known as d-achromatic number denoted by $\psi^d(G)$. Also we have $\chi^d(G) \le \psi^d(G) \le \psi_s^d(G)$ where $\chi^d(G)$ is the minimum number of colors required for proper complete d-coloring. In this paper, I found the exact values of this number $\psi_s^d(G)$ for a variety of family of graphs and defined this pseudo d-chromatic number in terms of partition graphs and investigate further results about the effect of removing points and lines on this number and upper bounds for this pseudo d-chromatic number.

The following results are proved in [4]

Proposition 1.1[4]: Let G be a k –pseudo complete d-colorable graph. Then $|V(G)| \ge k\left\lceil\frac{k-2d+1}{\Delta}\right\rceil$

Corollary 1.2[4]: For any graph,with maximum degree $\Delta, \psi_s^d(G) \le \max\{k / k\left\lceil\frac{k-2d+1}{2}\right\rceil \le |V(G)|\}$

Remark 1.3[4]: Let k and d be positive integer with $k \ge 2d$. Consider the graph $G_k^d = (V,E)$where $V = \{1,2,3,\ldots,k\}$ and $E(G) = \{(i,j)/D_k(i,j) \ge d\}$. Clearly G_k^d is $(k-2d+1)$ regular and $G_k^d$is k –pseudo complete d-colorable graph and size of the graph is $\frac{k(k-2d+1)}{2}$.

Proposition 1.4[4]: Let k and d be positive integers with $k \geq 2d$. Let $n(k,d)$ denote the integer $\frac{k(k-2d+1)}{2}$ or $\frac{k(k-2d+1)}{2}+\frac{k}{2}$ according as k is odd or even. Then the cycle on $n(k,d)$ vertices is k −pseudo complete d-colorable.

Proposition 1.5[4]: Let k and d be positive integers with $k \geq 2d$.

Let $n(k,d)=\begin{cases} \frac{k(k-2d+1)}{2}+1 \; if \; k \; is \; odd \\ \frac{k(k-2d+1)}{2}+\frac{k}{2} \; if \; k \; is \; even \end{cases}$

Then any path on $n(k,d)$ vertices is k-pseudo complete d-colorable.

2. Main results:

Theorem 1.1: Let n and d be positive integers with $n > d$, then $\psi_s^d(K_{n,n}) = n + d$ where $K_{n,n}$ is a complete bipartite graph.

Proof: Let $X = \{x_i\}_{i=1}^n$ and $Y = \{y_i\}_{i=1}^n$ be a bipartition of $K_{n,n}$. Consider the function $f: V(K_{n,n}) \rightarrow \{1,2,3,..,n,..,n+d\}$ defined by $f(x_i) = i$ and $f(y_i) = i + d, 1 \leq i \leq n$ gives a $(n+d)$ −pseudo complete d-coloring of G so that $\psi_s^d(K_{n,n}) \geq n + d$ (1)

By Corollary 1.2, we have $\psi_s^d(K_{n,n}) \leq \max\{n + d / \left\lceil \frac{n-d+1}{n} \right\rceil (n+d) \leq V(K_{n,n})\}$ so that

$\psi_s^d(K_{n,n}) \leq n + d$ —— (2)

Hence by (1) and (2) $\psi_s^d(K_{n,n}) = n + d$.

Remark 1.2: If a graph G admits a k −pseudo complete d-coloring then for any pair of colors $i, j \in Z_k$ with $D_k(i,j) \geq d$, there exists atleast one edge whose end vertices receive the colors i and j. Hence $|E(G)| \geq \frac{k(k-2d+1)}{2}$.

Theorem 1.3: The Jelly fish graph $J(m,n)$ is a 5-pseudo complete 2-colorable graph and $\psi_s^2(J_{m,n}) = 5$.

Proof: The graph $J_{m,n}$ is defined as follows:

$V(J_{m,n}) = \{u,v,w,x\} \cup \{u_i / 1 \leq i \leq m\} \cup \{v_i / 1 \leq j \leq n\}$ and

$$E(J_{m,n}) = \{(u,u_i)/1 \le i \le m\} \cup \{(v,v_j)/1 \le j \le n\} \cup \{(u,w),(u,x),(w,v),(w,x),(x,v)\}$$

Define $C: V(J_{m,n}) \to \{1,2,3,4,5\}$ as follows:

$C(u) = 2, C(w) = 4, C(x) = 1, C(v) = 3, C(u_1) = 5, C(v_1) = 5$ and assign for the rest of the vertices any of these five colors. Then C is 5-pseudo complete 2-coloring of $J_{m,n}$ and hence $\psi_s^2(J_{m,n}) = 5$. Claim: $\psi_s^2(J_{m,n}) \le 5$. Suppose $\psi_s^2(J_{m,n}) = 6$ under some optimal pseudo achromatic d-coloring obviously f will assign distinct colors to the higher degree vertices. Therefore the colors of u, v, w, x must be distinct. Now the 6th color must appear on the pendent vertices of $J_{m,n}$. Since x and w are not adjacent to the pendent vertices there exists atleast one color pair with distance d having no edge between them in $J_{m,n}$, a contradiction. Hence $\psi_s^2(J_{m,n}) \le 5$. Therefore $\psi_s^2(J_{m,n}) = 5$.

The figure below shows the 5-pseudo complete 2-coloring of the Jelly fish graph $J_{5,4}$

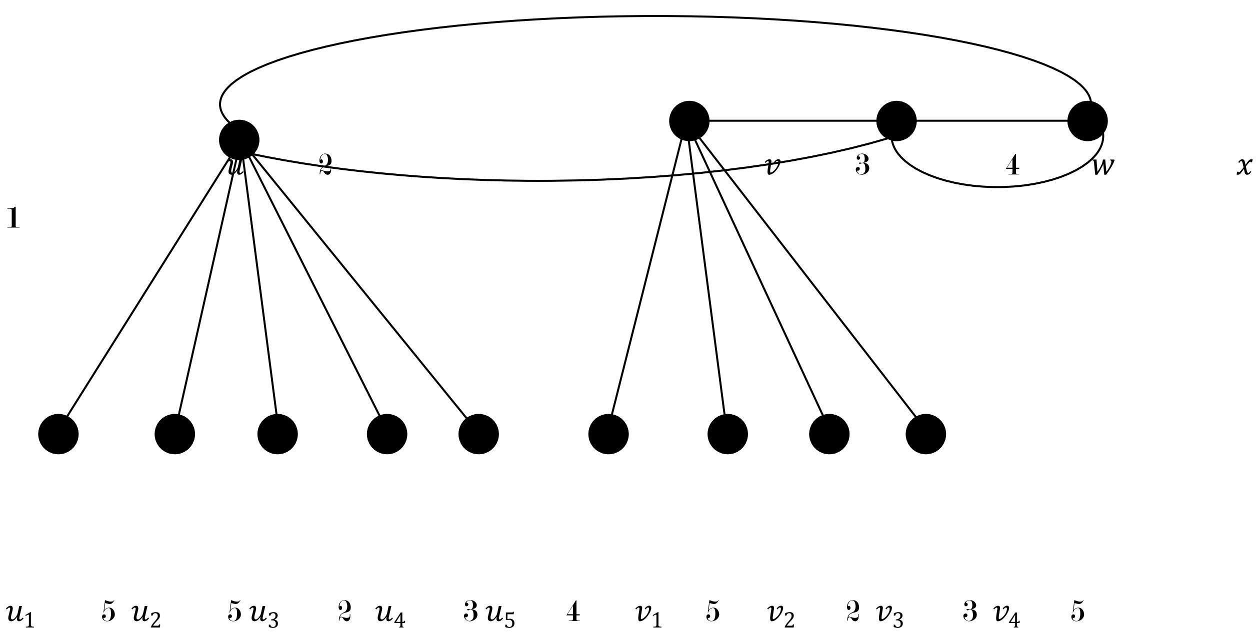

Figure (1)

Theorem 1.4: Let G be any graph. Then the 1 crown graph $GOK_{1,1}$ obtained from G by identifying the central vertex of $K_{1,1}$ with each vertex of G. Then $\psi_s^d(GOK_{1,1}) = \psi + 1$.

Proof: Let $V(G) = \{u_1, u_2, \dots u_p\}$. Let $V(G \odot K_{1,1}) = V(G) \cup \{u_1', u_2', \dots . u_p'\}$ and $E(G \odot K_{1,1}) = E(G) \cup \{(u_i, u_i')/1 \leq i \leq p\}$. Let f be any pseudo complete d-coloring of $G \odot K_{1,1}$. Let $f(V(G \odot K_{1,1}) = f(V(G) = \{1,2,.., \psi\}$ and let the set of vertices in $\{u_1', u_2', \dots . u_p'\}$ which are adjacent to the vertices of G receiving colors 1,2,..,d-1 assigned colors $\psi - (d-1) + i$ where $1 \leq i \leq d-1$ and the remaining vertices of $\{u_1', u_2', \dots . u_p'\}$ assigned the color $\psi + 1$under f. Then f is a $\psi + 1$ −pseudo complete d-coloring of $G \odot K_{1,1}$ and hence $\psi_s^d(G \odot K_{1,1}) \geq \psi + 1$. We claim $\psi_s^d(G \odot K_{1,1}) = \psi + 1$. Suppose f uses $\psi + 2$ colors then $(\psi + 2)^{th}$ color has to be adjacent with the ψcolors of G and the set of vertices u_i' receiving the $(\psi + 1)^{th}$ color with the distance d . Suppose f has a few redundant vertices where one of the ψ-colors is used. We can recolor any such vertex with the $(\psi + 2)^{th}$ color. But as the maximum number of colors used in a pseudo complete d-coloring of G is ψ, $(\psi + 2)^{th}$ color must have non adjacency with some of the ψ-colors of G. Those colors cannot be assigned to the vertices u_i' and hence results in atleast one pair of colors in which $(\psi + 2)^{th}$ color is present with no edge between them, a contradiction. Therefore $|f| \leq \psi + 1$ and hence $\psi_s^d(G \odot K_{1,1}) = \psi + 1$.

Theorem 1.5: Let k and$d$ be positive integers with $k \geq 2d$. Let C_n be the cycle on n vertices. Then $\psi_s^d(C_n) = 2k - 1, 2k$ or $2k + 1$ according as :$(k - d)(2k - 1) \leq n \leq k(2k - 2d + 1) + k - 1$ or $k(2k - 2d + 1) + k \leq n \leq (k - d + 1)(2k + 1) - 1$ or $(k - d + 1)(2k + 1) \leq n \leq (k - d + 2)\ (2k + 1) - 1$.

Proof: First observe that, for any n, d there is a unique k such that $(k - d)(2k - 1) \leq n \leq (k - d + 2)(2k + 1) - 1$. If $\psi_s^d(C_n) = \psi$,then C_n contains atleast $\frac{\psi(\psi - 2d + 1)}{2}$ edges and hence $\frac{\psi(\psi - 2d + 1)}{2} \leq n$. Therefore $\psi_s^d(C_n) \leq 2k$ if $(k - d)(2k - 1) \leq n \leq (k - d + 1)(2k + 1) - 1$ and $2k + 1$ if $(k - d + 1)(2k + 1) \leq n \leq (k - d + 2)(2k + 1) - 1$

Case (i): $(k - d)(2k - 1) \leq n \leq k(2k - 2d + 1) + k - 1$

Clearly $\psi_s^d(C_n) \leq 2k$ suppose $\psi_s^d(C_n) = 2k$. Let f be any $2k$ −pseudo complete d-coloring of C_n with color classes $V_1, V_2, \dots V_{2k}$ where for $1 \leq i \leq 2k, V_i$ be the set of vertices, receiving color c_i. Obtain a new graph G^* with vertex set $v_1, v_2, \dots v_{2k}$. The edge set $E(G^*)$ is obtained by introducing S_{ij} edges joining v_i and v_j where $S_{ij} = |[V_i, V_j]|$ with $D_k(i, j) \geq d$ is the number of edges of C_n having one end in V_i and other end in V_j. As f is $2k$ −pseudo d-complete, $S_{ij} \geq 1$ for each $i \neq j$. Clearly G^* is Euclerian and an Euler Tour of G^* can be obtained by traversing in the order of the vertices of C_n also f is $2k$ −psuedo d-complete, the underlying graph of G^* is G_{2k}^d. To obtain an Eulerian super graph of the odd,regular graph G_{2k}^d we have to add atleast k new edges. Hence $n \geq k(2k-2d+1)+k$, a contradiction. Hence $\psi_s^d(C_n) \leq 2k-1$. To obtain equality, we have to obtain a $(2k-1)$ −pseudo complete d- coloring for C_n. For this consider G_{2k-1}^d and label its vertices by $v_1, v_2, \dots v_{2k-1}$. Let T be any euler tour of G_{2k-1}^d. If the i^{th} edge of T is say $v_k v_l$ then color the i^{th} vertex of $C_{(k-d)(2k-1)}$ by c_l. This yields a $(2k-1)$ −pseudo complete d-coloring of $C_{(k-d)(2k-1)}$ with $(2k-1)$ −colors. Now this coloring of $C_{(k-d)(2k-1)}$ can be extended to $(2k-1)$ −pseudo complete d-coloring of C_n by subdividing an edge $e=(u,v)$ of $C_{(k-d)(2k-1)}, n - C_{(k-d)(2k-1)}$ times and assigning to each new vertex either the color of u or the color of v. Hence $\psi_s^d(C_n) = 2k-1$.

Case 2: $k(2k-2d+1)+k \leq n \leq (k-d+1)(2k+1)-1$

Once again $\psi_s^d(C_n) \leq 2k$. To establish the equality, consider a perfect matching F of G_{2k}^d and obtain a new graph $G_{2k}^{d^*}$ from G_{2k}^d by duplicating the edges of F. Clearly $G_{2k}^{d^*}$ is $(2k-2d+2$ −regular and hence it is eulerian. Let T be the euler tour of $G2kd*$. As in case (i), T gives $2k$-pseudo complete d-coloring for the cycle $C_{k(2k-2d+1)+k}$. Using the subdivision method as in case (i), we can obtain a $2k$ −pseudo complete d-coloring for C_n.

Case (iii) $(k-d+1)(2k+1) \leq n \leq (k-d+1)(2k+1)-1$

Here consider the graph G_{2k+1}^d and its Euler tour T and proceed as in case (i).

The figure (2) below shows $G_6^{2^*}$ from G_6^2 by duplicating the edges $(1,4),(2,5),(3,6)$ which $2k-2d+2=$ 4-regular.

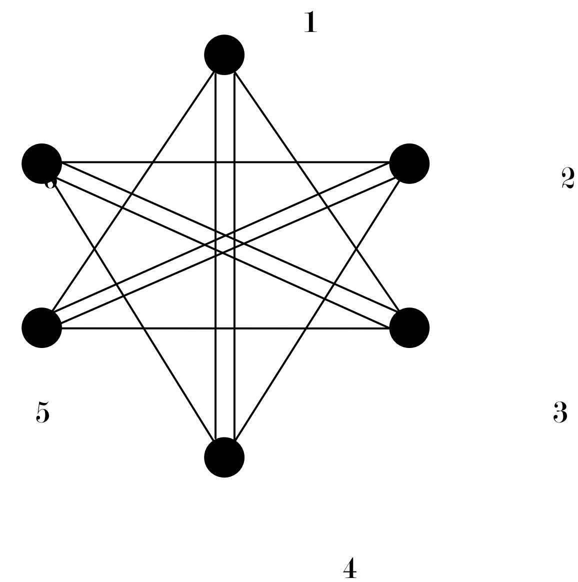

Figure (2)

Corollary 1.6: Let P_n be the path of length $n-1$. Then $\psi_s^d(P_n) = 2k-1, 2k$ or $2k+1$ according as $(k-d)(2k-1) \leq n \leq k(2k-2d+1)-1, k(2k-2d+1) \leq n \leq (k-d+1)(2k+1)-1$ or $(k-d+1)(2k+1) \leq n \leq (k-d+2)(2k+1)-1$. This can be proved on similar lines as in theorem 1.5.

Theorem 1.7: Let $n_1, n_2, \ldots., n_m, d$ be positive integers, where $m \geq 3$, $n_1 \leq n_2 \leq \ldots. \leq n_m$ and $n+(m-1)d+1 \geq 4d$. Also let $n_m \leq \frac{n-(m-1)d+1}{2}+1$ where $n = \sum n_i$. Then $K_{n_1,n_2,\ldots,n_m}$ is $\left\lfloor \frac{n+(m-1)d+1}{2} \right\rfloor$ −pseudo complete d-colorable graph and $\psi_s^d\left(K_{n_1,n_2,\ldots,n_m}\right) = \left\lfloor \frac{n+(m-1)d+1}{2} \right\rfloor$.

Proof: Let $V_i = \{v_{ij}/1 \leq j \leq n_i\}$ denote i^{th} partite set of $K^* = K_{n_1,n_2,\ldots,n_m}$. For $1 \leq i \leq m$, color v_{i1} by $c_{(i-1)d+1}$. Now arrange the remaining vertices $V' = V(K^*) - \{v_{i1}/1 \leq i \leq m\}$ in the lexicographic order with respect to the indices i, j of v_{ij}. If n and m are of the same parity, then color the $n-m$ vertices of V' with colors $c_2, c_3, \ldots c_d, c_{d+1}, \ldots c_{\frac{n+(m-1)d+1}{2}}$, $c_2, c_3, \ldots c_d, \ldots$ respectively in order. This color will be pseudo d- complete if and only if there is no positive integer j,where $md+1 \leq j \leq \frac{n+(m-1)d+1}{2}$ such that the set of the vertices of V' corresponding to the sequence $c_j, c_{j+1}, \ldots c_{\frac{n+(m-1)d+1}{2}}$, $c_{md+1}, \ldots$ c_j is contained in some partite set of K^*. But the cardinality of such a set is $\frac{n+(m-1)d+1}{2}+(1-j)+(j-md)=\frac{n+(m-1)d+1}{2}+1$ and hence taking into account, the initial vertex of this partite set, this partite set should contain $\frac{n+(m-1)d+1}{2}+2$ vertices. But this is not the case as per our hypothesis. Hence the given coloring is a pseudo complete d-coloring of K^*using $\left\lfloor\frac{n+(m-1)d+1}{2}\right\rfloor$colors when n and m are of same parity. If n and m are of opposite parity, color the vertices of V' with $c_{md+1},\ldots, \ldots c_{\frac{n+(m-1)d}{2}}, c_{md+1}, \ldots, \ldots c_{\frac{n+(m-1)d}{1}}, c_{md+1}, \ldots$ respectively in order. Once again this coloring yields a pseudo complete d-coloring of K^* using $\left\lfloor\frac{n+(m-1)d+1}{2}\right\rfloor$ colors. Hence $\psi_s^d(K^*)\geq \left\lfloor\frac{n+(m-1)d+1}{2}\right\rfloor$ To establish equality, consider any pseudo complete d-coloring ς of K^* with $\psi_s^d(K^*)=\psi$. Let $C_1=\{c_i/c_i$ is assigned to exactly one vertex of $K^*\}$and $C_2=\varsigma(V(K^*)-C_1$. Let $|C_i|=x_i$ for $i=1,2$. Then we have $x_1\leq(m-1)d+1$ and $x_1+x_2=\psi$. If $\psi\geq\left\lfloor\frac{n+(m-1)d+1}{2}\right\rfloor$, $x_1+2x_2\leq n$ and hence $2\psi\leq x_1+n$. Suppose $\psi\geq\frac{n+(m-1)d+2}{2}$ which implies $n+(m-1)d+2\leq x_1+n$ then $(m-1)d+2\leq x_1$, a contrtadiction. Therefore $\psi_s^d(K^*)=\left\lfloor\frac{n+(m-1)d+1}{2}\right\rfloor$.

2. Further results of pseudo-d-achromatic number using Partition graphs

The chromatic, achromatic and pseudo achromatic numbers were defined by Sampath Kumar and Bhave [5] in terms of the partition graphs. Now we define d-chromatic number $\chi^d(G)$, d-achromatic number $\psi^d(G)$ and pseudo d-achromatic number $\psi_s^d(G)$ in terms of Partition graphs.

Definition 2.1: Let P be a partition of V(G) of a graph G. The Partition graph P(G) of G is a graph with point set $P=\{V_1,V_2,\ldots V_k\}$ where V_i and V_j are adjacent if there exists $v_i\in V_i$ and $v_j\in V_j$ such that v_iv_j is a line in G.

Definition 2.2: Let k and d be positive integers with $k\geq 2d$. A partition is complete with respect to (k,d) −coloring if $P(G)=G_k^d$. Let $p(G)$ denote the class of all partition graphs of G, and $\overline{p(G)}$ denote the class of all partition Graphs of G which are homomorphic images of G. It is easy to see that

Lemma 2.3: For a graph G, n and d be positive integers with $n\geq 2d$.

1) $\chi^d(G) = min\{n/G_n^d \in \overline{p(G)}\}$

2) $\psi^d(G) = max\{n/G_n^d \in \overline{p(G)}\}$

3) $\psi_s^d(G) = max\{n/G_n^d \in p(G)\}$ whereas $\psi_s(G) = max\{n/K_n \in p(G)\}$

Theorem 2.4: For any graph G and any point $u \in V(G), \psi_s^d(G) \geq \psi_s^d(G-u) \geq \psi_s^d(G) - 1$

Proof: Let $\psi_s^d(G-u) = n$. Let n, d be positive integers with $n \geq 2d$. Then there exists a partition $P' = \{V_1', V_2', \dots V_n'\}$ with respect to d-coloring of $V(G) - \{u\}$ such that $P'(G - \{u\}) = G_n^d$. Now since $P = \{V_1' \cup \{u\}, V_2', \dots, V_n'\}$ is a partition of V(G) such that $P(G) = G_n^d$. It follows that $\psi_s^d(G) \geq \psi_s^d(G-u)$. To prove the other inequality, suppose $\psi_s^d(G) = n$, then there exists a partition P of V(G) with respect to d-coloring such that $P(G) = G_n^d$. Let $u \in V_1 \in P$. Clearly the points of $P - \{V_1\}$ induces G_{n-1}^d as subgraph in P(G) and hence $\psi_s^d(G-u) \geq n-1 = \psi_s^d(G) - 1$. Thus we get $\psi_s^d(G) \geq \psi_s^d(G-u) \geq \psi_s^d(G) - 1$.

Theorem 2.5: For any graph and a line $e \in E(G), \psi_s^d(G) \geq \psi_s^d(G-e) \geq \psi_s^d(G) - 1$

Proof: Suppose $\psi_s^d(G) = n$. Let n, d be positive integers with $n \geq 2d$. Then there exists a partition P of V(G) such that $P(G) = G_n^d$ with respect to d-coloring of G. If the line e joins a point of V_i to a point of V_j where $V_i, V_j \in P, i \neq j$ then the partition $P' = P - \{V_i, V_j\} \cup \{V_i \cup V_j\}$ of V(G) will be such that $P'(G - \{e\})$ which induces G_{n-1}^d as subgraph. Hence $\psi_s^d(G-e) \geq n - 1 = \psi_s^d(G) - 1$ on the other hand, if P is a partition of V(G) such that $P'(G - \{e\}) = G_n^d$ with respect to d-coloring of G where $m = \psi_s^d(G-e)$then for the same partition P, $P(G) = m$ and hence $\psi_s^d(G) \geq m = \psi_s^d(G-e)$.

Theorem 2.6: For any graph G, with $\psi_s^d(G) > \psi^d(G)$ there exists a line x such that $\psi_s^d(G-$ *x=ψsdG*

Proof: Let $\psi_s^d(G) > \psi^d(G)$ and $\psi_s^d(G) = n$ where n, d are positive integers with $n \geq 2d$. Then there exists a partition P of V(G) such that $P(G) = G_n^d$ with respect to d-coloring of G. Clearly the partition is not homomorphic, for otherewise $\psi_s^d(G) \geq n$. Hence there is a line joining points of the same set V_i of P. This line x will be such that $P(G-x) = G_n^d$. Therefore $\psi_s^d(G-$ *x≥n* but by theorem 2.5, *ψsdG−x≤n.* Thus *ψsdG−x=n=ψsdG.*

Remark 2.7: The converse of theorem 2.6 need not be true. The graph G in figure (3) has a line x such that $\psi_s^d(G-x) = \psi_s^d(G)$

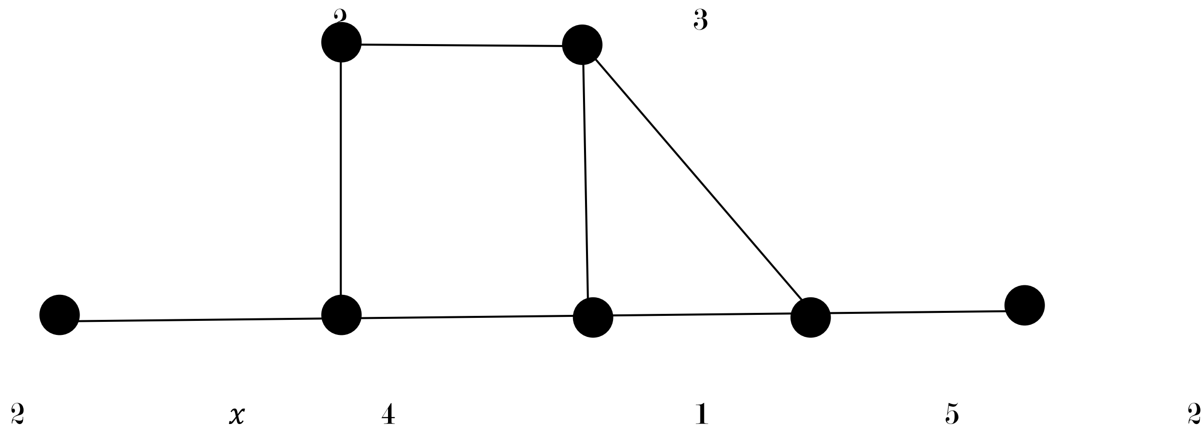

Figure (3)

Here $\psi_s^2(G) = \psi_s^2(G-x) = 5$. But $\psi_s^2(G) = \psi^2(G) = 5$.

Theorem 2.8: If G is a graph with q lines then $\psi^d(G) \leq \psi_s^d(G) \leq r$ where r is the maximum integer with $q \geq \frac{r(r-2d+1)}{2}$.

Proof: Let $\psi_s^d(G) = n$ where n and d are positive integers with $n \geq 2d$. Then there exists a partition P of V(G) such that $P(G) = G_n^d$ with respect to any d-coloring of G. Then G has atleast $\frac{n(n-2d+1)}{2}$ lines. Hence $q \geq \frac{n(n-2d+1)}{2}$. If $G = qK_2$ where $q = \frac{r(r-2d+1)}{2}$ then $\psi_s^d(G) = r$. This shows that the bound is attained.

Definition 2.9: A graph H is a partition realizable from a graph G, if P(G) = H for some partition P of V(G).

Lemma 2.10: If G is a graph with q lines and no isolated points then G is a partition realizable from qK_2

that is a graph with q copies K_2.

Lemma 2.11: If H is a subgraph of G, $q$andq_1 are the number of lines in G and H respectively, then G is a partition realizable from $H \cup rK_2$ where $r = q - q_1$. In lemma 2.10 and 2.11, in forming the partition graphs we observe that no lines of G are destroyed and hence the partitions are homomorphisms of G.

Theorem 2.12: If a and b $(a > b > 2d)$ are two positive integers then there exists a graph G with $\psi^d(G) = \psi_s^d(G) = a$ and $\chi^d(G) = b$

Proof: Let $G = G_b^d \cup rK_2$ where $r = \frac{a(a-2d+1)}{2} - \frac{b(b-2d+1)}{2}$. Then by lemma 2.11, there exists a partition P of V(G) with $P(G) = G_a^d$. Hence with respect to any d-coloring of G, $\psi_s^d(G) \geq a$. Also G has $\frac{a(a-2d+1)}{2}$ lines. Hence $\psi_s^d(G) \leq a$. Thus $\psi_s^d(G) = a$.Since P is a homomorphism of G, we have $\psi^d(G) = a$ and $\chi^d(G) = b$ follows since G_b^d is a component of G and every other component of G is K_2. Hence the proof.

Definition 2.13: A graph G is k-achro-d-critical if $\psi^d(G - x) < \psi^d(G) = k$ for every line x in G where $k \geq 2d$.

Theorem 2.13: If G is k -achro d-critical where $k \geq 2d$ then G has exactly $\frac{k(k-2d+1)}{2}$ lines and if $\psi^d(G) = k$ and G has $\frac{k(k-2d+1)}{2}$ lines then G is k-achro d- critical .

Proof: Let G be k -achro d-critical graph, let $P(G) = G_k^d$ where $P = \{V_1, V_2, \dots V_k\}$ is a homomorphism with respect to d-coloring of G. Then each set V_i is independent and there is only one line joining V_i and V_j in G with $D_k(i,j) \geq d$ where $i, j \in \{1,2,..,k\}, i \neq j$. This implies G has exactly $\frac{k(k-2d+1)}{2}$ lines. Conversely, if $\psi^d(G) = k$ and G has exactly $\frac{k(k-2d+1)}{2}$ lines then for any line x in G we have $\psi^d(G - x) < k$ by theorem 2.8. Hence G is k -achro d-critical.

Theorem 2.14: A graph G is k -achro d-critical if and only if it is k -pseudo d-critical (that is k edged d-critical)

Proof: Let G be k -pseudo d-critical then there exists a complete partition $P = (V_1, V_2, \dots V_k)$ of V(G) such that $P(G) = G_k^d$ with respect to d-coloring of G. We claim P is a homomorphism of G. For suppose $V_i \in P$ is not independent and x be a line of G with both ends in V_i, then for some partition P of V(G), we get $P(G - x) = G_k^d$. This implies that $\psi_s^d(G - x) \geq k$, which is a contradiction. Thus P is a homomorphism of G. Hence $\psi^d(G) \geq k$, but $\psi^d(G) \leq \psi_s^d(G) = k$ which implies $\psi^d(G) = k$. Also for every line x of G, $\psi^d(G - x) \leq \psi_s^d(G - x) < \psi_s^d(G) = k$

(since G is k pseudo d-critical) . Therefore we get $\psi^d(G-x)<k$. Hence G is k -achro d-critical. Conversely, suppose G is k -achro d-critical. By theorem 2.13, G has $\frac{k(k-2d+1)}{2}$ lines. Hence by theorem 2.8, $\psi_s^d(G)\le k=\psi^d(G)$. But we have $k=\psi^d(G)\le\psi_s^d(G)$. Hence $\psi_s^d(G)=k$. Again by theorem 2.8, $\psi_s^d(G-x)<k$ for any line x. Hence G is k -pseudo d-critical.

Definition 2.15: An k -edge d-critical graph is one which is k -pseudo d-critical (hence k -achro d-critical).

Construction of k -edge d-critical graphs:

Let H be a subgraph of G. Then we shall denote the subgraph of G obtained by deleting all lines of H and the resulting isolated points in G by $G-H$. It is clear that

Lemma 2.16: Let a graph H be partition realizable from G and H_1 be an induced subgraph of H, then 1) H_1 is a partition realizable from an induced subgraph G_1 of G. 2) $H-H_1$ is a partition realizable from $G-G_1$.

We observe that $G_k^d-G_{k-1}^d=K_{1,k-2d+1}\cup(d-1)K_2$ where $k\ge 2d$.

Corollary 2.17: If G_k^d where $k\ge 2d$ is a partition realizable from G, then there exists an induced subgraph G_1 of G such that

(i) G_{k-1}^d is a partition realizable from G_1 and

(ii) $K_{1,k-2d+1}\cup(d-1)K_2$ is a partition realizable from $G-G_1$.

The above lemma suggests a method of constructing k -edge d-critical graphs from the set of all $(k-1)$ −edge d-critical graphs. The method is as follows. We consider the graphs with no isolated points.

Let $\{G_i\}$ be the collection of all $(k-1)$ −edge d-critical graphs and $\{H_j\}$ be the collection of all graphs with $k-2d+1+d-1=k-d$ lines such that $K_{1,k-2d+1}\cup(d-1)K_2$ is a partition realizable from H_j. Since G_{k-1}^d is a partition realizable from G_i and $K_{1,k-2d+1}\cup(d-1)K_2$ is a partition realizable from H_j, G_k^d is a partition realizable from G each of the graphs formed below. Further each of the following graphs has exactly $\frac{k(k-2d+1)}{2}$ lines. Hence each is k -edge d-critical graph.

Consider a graph G_i. Let $P = \{V_1, V_2, \dots V_{k-1}\}$ be a complete partition of V(G_i). It is easy to see that each H_j has atleast $k - 2d + 1 + 2(d-1) = k - 1$ points of degree 1 say u_r where $r = 1,2, \dots, k-1$. Let G be a graph obtained from G_i and H_j by identifying some, all or none of the points u_r with the points of G_i such that no two points u_r are identified with the points of same set $V_i \in P$. We claim that any k-edge d-critical graph is isomorphic to a graph obtained above. For, let G be k -edge d-critical graph and $P(G) = G_k^d$. Then as G_{k-1}^d is an induced subgraph of G_k^d, there exists an induced subgraph $G_i^{'}$ of G such that G_{k-1}^d is a partition realizable of $G_i^{'}$ and $K_{1,k-2d+1} \cup (d-1)K_2$ is a partition realizable from $G - G_i^{'}$ by corollary 2.17. Therefore $G_i^{'}$ has $\frac{(k-1)(k-2d)}{2}$ lines and hence it is $(k-1)$ −edge d- critical and $G - G_i^{'}$ has k -d lines. Therefore G is isomorphic to one of the graphs obtained above.

3. Some upper bounds of $\psi_s^d(G)$:

Let $\beta_0, \beta, \alpha_0, \alpha$ denotes the point independent number, line independent number, point covering number, line covering number respectively.

Theorem 3.1: For any graph G with p points, $\psi_s^d(G) \leq p - \beta_0 + 2d - 1$.

Proof: Let $\psi_s^d(G) = r$. Clearly there exists a partition $P = \{V_1, V_2, \dots V_r\}$ of V(G) such that $P(G) = G_r^d$ where $r \geq 2d$ with respect to d-coloring of G. Let S be the set of β_0 independent points of G. Since any two $V_i, V_j, i \neq j$ are adjacent in P(G), with $\left|V_i - V_j\right|_r \geq d$. It can be seen that $V_i \cup V_j$ is not contained in S for all i, j. This implies atleast $r - 2d + 1$ of the sets in P intersect $V(G) - S$. Thus,

$$r - 2d + 1 \leq |V(G) - S| \leq p - \beta_0$$

$$\chi_s^d = r \leq p - \beta_0 + 2d - 1$$

Hence $\chi_s^d \leq p - \beta_0 + 2d - 1$

Corollary 3.2: For any graph G, $\chi_s^d(G) \leq \alpha_0 + 2d - 1$ where α_0 is the point covering number of G.

Proof: From theorem 3.1, we get $\chi_s^d(G) \leq p - \beta_0 + 2d - 1$ ———— (1) where β_0 is the point independence number of a graph G. Already, we know that $\alpha_0 + \beta_0 = p$ where α_0 is the point covering number of G. Hence $\beta_0 = p - \alpha_0$. Applying (1) we get $\chi_s^d(G) \leq p - (p - \alpha_0) + 2d - 1$. Therefore $\chi_s^d(G) \leq \alpha_0 + 2d - 1$.

Theorem 3.3: For any graph G, (i) $\psi_s^d(G) \leq 2\beta_1 + 2d - 1$ and (ii) $\psi_s^d(G) \leq 2(p - \alpha_1 + d) - 1$ where β_1, α_1 be the line independence number and line covering number respectively.

Proof: (i) If $\psi_s^d(G) = n$ where $n \geq 2d$ then there exists a partition P of V(G) such that $P(G) = G_n^d$ with respect to any d-coloring of G. Also we have $\beta_1(G_n^d) = \frac{1}{2}(n - 2d + 2)$ or $\frac{1}{2}(n - 2d + 1$ according as n is even or odd. Hence we get, *n≤2β1Gnd+2d−1* which implies $n \leq 2\beta_1\big(P(G)\big) + 2d - 1$. Therefore $n \leq 2\beta_1(G) + 2d - 1$. Hence $\psi_s^d(G) \leq 2\beta_1 + 2d - 1$.

(ii)Moreover we have $\alpha_1 + \beta_1 = p = \alpha_0 + \beta_0$. Therefore $\beta_1 = p - \alpha_1$ where α_1 is the line covering number of G. Substituting in (i) we get $\psi_s^d(G) \leq 2(p - \alpha_1) + 2d - 1$. Hence $\psi_s^d(G) \leq 2(p - \alpha_1 + d) - 1$

4. Existence of graphs with the given pseudo-d-achromatic number

Theorem 4.1: For any positive integers m,n,d such that $m > n$ and $m + n \geq d$ there exists a graph whose pseudo-d-achromatic number is $m + n + d + 1$.

Proof: Construct a graph and call it $G_{m,n}$ which is a bipartite graph with a complete bipartite subgraph. This graph has bipartition (A, B) where A is $\{u_1, u_2, \dots, u_m\} \cup \{y_1, y_2, .. y_n\}$ and B is $\{v_1, v_2, \dots, v_m\} \cup \{x_1, x_2, .. x_n\}$. $E(G) = \{(u_i, x_j)/1 \leq i \leq m, 1 \leq j \leq n\} \cup \{(v_i, y_j)/1 \leq i \leq$ *m,1≤j≤nUyi,xj/1≤i≤n,1≤j≤nUui,vj/1≤i≤j≤mUu,v2,v,yn.*

Lower bound: Let us color the graph $G_{m,n}$ as follows:

- For $1 \leq i \leq m$, color u_i with c_i
- For $1 \leq i \leq m$, color v_i with c_{i+d}
- For $1 \leq j \leq n$, color x_i with c_{m+j+d}
- For $1 \leq j \leq n$, color y_i with $c_{m+j+d+1}$
- Color u with c_{m+d+1}
- Color v with c_d

This yields a pseudo complete d-coloring of $G_{m,n}$ with $m+n+d+1$ colors. Therefore $\psi_s^d(G_{m,n}) \geq m+n+d+1$

Upper bound:

$G_{m,n}$ is a subgraph of $K_{m+n+1,m+n+1}$. Let f be a pseudo complete d-coloring of $K_{n,n}$. Assume $|f(V(K_{n,n})| \geq n+d+1$ where $d < n$, this means there exists $d+1$ colors, which are not represented in one part (an independent set of vertices) of the graph. This means they must be represented in other part which is also an independent set as it is a bipartite graph. Thus there are no two vertices colored with $c_1, c_2, .., c_{d+1}$ that are adjacent. This is a contradiction to it being a pseudo complete d-coloring of $K_{n,n}$. Hence $\psi_s^d(K_{n,n}) < n+d+1$. This means $\psi_s^d(K_{m+n+1,m+n+1}) \leq m+n+d+1$. Hence $\psi_s^d(G_{m,n}) \leq m+n+d+1$. Therefore $\psi_s^d(G_{m,n}) \leq m+n+d+1$.

The graph $G_{4,3}$ is pictured in the figure below with the double line meaning that the two set of vertices are joined that is, every vertex in one subset is adjacent to every vertex in other subset. Here $\psi_s^d(G_{4,3}) = 11$

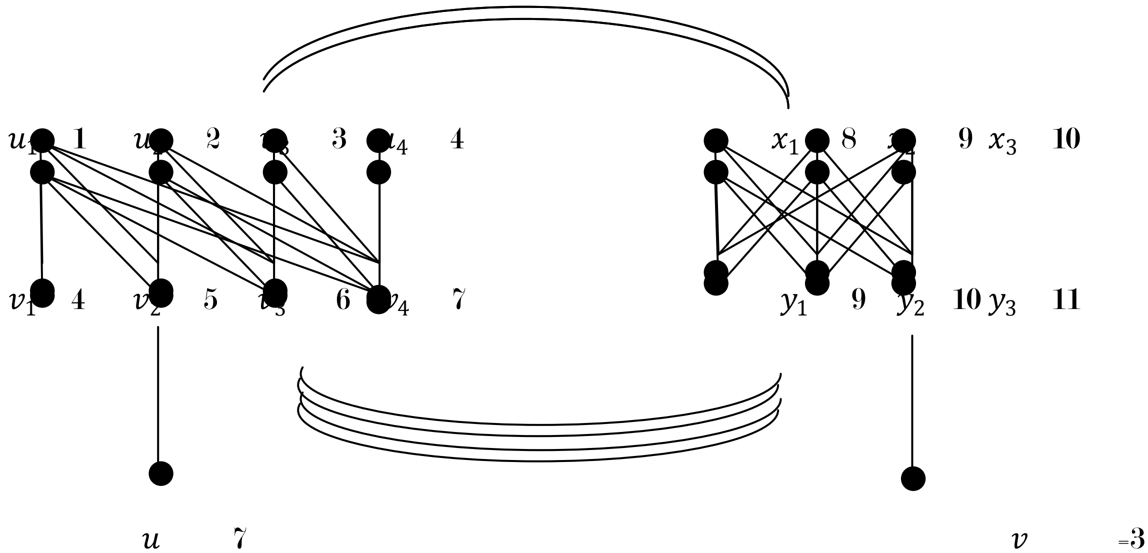

Figure (4)

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