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THE EXISTENCE OF POSITIVE SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATION WITH BOUNDER VALUE PROBLEM IN THE BANACH SPACE.

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ABSTRACT

In this paper, we have shown the existence of positive solution of the equation $u'' + r(t)f(t,u) = 0$ *with the boundary conditions. We had showed the existence of at least one positive solution based on if a function* f *is either superlinear or sublinear by simple application of a fixed point theorem. The investigated solution in the classical Banach space of smooth functions* $C([0,\infty))$ *with regard to second order ordinary differential equation clearly explained in this paper.*

Keywords: Positive solutions, super linear, sub linear, boundary value problem and fixed point index.

1. INTRODUCTION

The theory of differential equation often times involves the theory of function spaces that are defined in terms of properties of pertinent functions and their derivatives. Differential equations modeled in Banach spaces have attracted the attention of many researchers throughout last century. Most of the efforts are concentrated in the study of the initial value problem. In this regard, we are going to investigate solution in the classical Banach space of smooth functions$C\big([0,\infty)\big)$. An ordinary differential equation arises in many different areas of applied mathematics and physics; see [9,11] for some references along this line. In this paper we would considered the second-order boundary value problem (BVP)

$$\boldsymbol{u}'' + \boldsymbol{r}(\boldsymbol{t})\boldsymbol{f}(\boldsymbol{u},\boldsymbol{t}) = \mathbf{0}\,,\ \ \mathbf{0} < t < 1 \qquad (1.1)$$

$$\begin{cases} \boldsymbol{\alpha u}(\mathbf{0}) - \boldsymbol{\beta u}'(\mathbf{0}) = \mathbf{0}, \\ \boldsymbol{\gamma u}(\mathbf{1}) + \boldsymbol{\delta u}'(\mathbf{1}) = \mathbf{0}. \end{cases} \quad \text{and where } \alpha,\beta,\gamma,\delta \geq 0 \qquad (\ \mathbf{1.2})$$

By the positive solution of (1.1), (1.2), it means that a function $\boldsymbol{u}(\boldsymbol{t})$ is positive on $0 < t < 1$ and satisfies differential equation (1.1).

And we consider two versions of assumptions that will provide different results.

(H.1) $f \in C([0,\infty),[0,\infty))$,

(H.2) $r \in C([0,1],[0,\infty))$ and $\boldsymbol{r}(\boldsymbol{t}) \neq \mathbf{0}$ **for** $t \in [0,1]$.

(H.3) $\alpha,\beta,\gamma,\delta \geq 0$ and let $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The boundary value problem (1.1) ,(1.2) arises in many different areas of applied mathematics and physics. Additional existence results may be found in [4, 7, 8, 10, 11] references.

2. Premilinaries

Theorem 2.1 Let E be a banach space, and let $K \subseteq E$ be a cone in E. Assume Ω_1,Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subseteq \Omega_2$, and let

$$R: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

i. $\|Ru\| \leq \|u\|\,, u \in K \cap \partial\Omega_1$, and $\|Ru\| \geq \|u\|\,, u \in K \cap \partial\Omega_2$; or

ii. $\|Ru\| \geq \|u\|, u \in K \cap \partial\Omega_1$, and $\|Ru\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then R has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$

Our purpose here is to give an existence result for positive solutions to the BVP (1.1) (1.2) assuming that f is either superlinear or sublinear. We do not require any monotonicity assumptions on f. To be precise, we introduce the notation

i. $f_0 \coloneqq \lim_{u\to 0} \frac{f(u)}{u}$
ii. $f_\infty \coloneqq \lim_{u\to\infty} \frac{f(u)}{u}$

Thus, $f_0 = 0$ and $f_\infty = \infty$ correspond to the suplinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By a positive solution of (1.1), (1.2) ,we understand a solution $u(t)$ which is positive on $0 < t < 1$ and satisfies the differential equation (1.1) for $0 < t < 1$ and the boundary conditions (1.2) .By a change of variable , the existence of a positive of (1.1), (1.2) may be shown to be equivalent to the existence of a positive radial solution of the semi linear elliptic equation $\Delta u + g(|x|)f(u) = 0$ in the annulus $R_1 < |x| < R_2$ subject to certain boundary conditions for $R_1 = |x|$ and $|x| = R_2$ we refer to [10] for some additional details.

Lemma 2.2 Let $\alpha\beta$= 1. Then, for $r \in C[0;\ 1]$, the boundary value problem (1.1) , (1.2) has the Unique solution .

The proof of (2.1) follows along the lines of the proof that is given in [7] and

Hence we omit it.

3. The Existence Results

The main result of this paper is to investigate at least one positive solution in the case of suplinear and sublinear case.

Theorem1: Assume (H.1)-(H.3) hold . Then the BVP (1.1), (1.2) has at least one positive solution in the either case.

i. $f_0 = 0$ and $f_\infty = \infty$ (sup linear) ,or
ii. $f_0 = \infty$ and $f_\infty = 0$ (sub linear)

It will be seen in the proof that theorem 1 is also valid for the more general equation

$$(1.1)^* \qquad u'' + f(t,u) = 0$$

with the boundary condition (1.2), provided we assume a certain uniformity with respect to the t variable. We state this more general result as:

Corollary 1: Assume f is continuous, $f(t,u) \geq 0$ for $t \in [0,1]$, and $u \geq 0$ with $f(t,u) \neq 0$ on any subinterval of [0,1] for $u > 0$; and let (H.3) hold. Then the boundary value problem (1.1)*, (1.2) has at least one positive solution in the either case.

a. $\lim_{u\to 0+} max_{t\in[0,1]} \frac{f(u)}{u} = 0$ and $\lim_{u\to\infty} max_{t\in[0,1]} \frac{f(u)}{u} = \infty$ or

b. $\lim_{u\to 0+} min_{t\in[0,1]} \frac{f(u)}{u} = \infty$ and $\lim_{u\to\infty} max_{t\in[0,1]} \frac{f(u)}{u} = 0$

The proof of theorem (1) will be based on application of the following fixed point theorem (theorem 2.1). The proof of the corollary (1) follows from the proof of theorem (1) with obvious slight modifications which we shall omit. We will apply the first and second parts of the above fixed point theorem to the superlinear and sublinear cases, respectively. We are going to see into ways .

<u>Case I</u> superlinear case i.e $f_0 = 0$ and $f_\infty = \infty$

Since (1.1) and (1.2) has a solution $u = u(t)$ iff u solves the operator equation

$$u(t) = \int_0^1 G(t,s)\, r(s) f\big(u(s)\big) ds \coloneqq Ru(t)$$

Where $G(t,s)$ denotes the green's function for the boundary value problem

$$u''(t) = 0 \qquad (2.1)$$

$$\alpha u(0) - \beta u'(0) = \gamma \qquad u(1) + \delta u'(1) = 0 \qquad (2.2)$$

And explicitly given by

$$G(t,s) = \begin{cases} \frac{1}{\rho}(\gamma + \delta - \gamma t)(\beta + \alpha s), 0 \leq s \leq t \leq 1 \\ \frac{1}{\rho}(\beta + \alpha t)(\gamma + \delta - \gamma s), 0 \leq t \leq s \leq 1 \end{cases}$$

Consider k be cone in $C[0,1]$ given by:

$$K = \left\{u \in C[0,1]: u(t) \geq 0, \min u(t) \geq M\|u\|, \frac{1}{4} \leq t \leq 3/4\right\} \qquad (2.3)$$

Where $\|u\| = \sup_{[0,1]}|u(t)|$ and $M = min\left\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\}$ (2.4)

Next, let us define

$$w(t) = (\gamma + \delta - \gamma t) \quad , \psi(t) = \beta + \alpha t \qquad (2.5)$$

So that

$$G(t,s) = \begin{cases} \frac{1}{\rho} w(t)\psi(s), 0 \leq s \leq t \leq 1 \\ \frac{1}{\rho} w(s)\psi(t), 0 \leq t \leq s \leq 1 \end{cases} \qquad (2.6)$$

In this we observe that $G(t,s) \leq w(s)\psi(s) = G(s,s),\ 0 \leq t \leq s \leq 1$

So that, if $u \in k$, then

$$Ru(t) = \int_0^1 G(t,s)r(s)f(u(s))ds \leq \int_0^1 G(s,s)r(s)f(u(s))ds \qquad (2.7)$$

Hence $\|Ru\| \leq \int_0^1 G(s,s)r(s)f(u(s))ds$ (2.8)

Furthermore, for $\frac{1}{4} \leq t \leq 3/4$

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{w(t)}{w(s)}, s \leq t \\ \frac{\psi(t)}{\psi(s)}, t \leq s \end{cases} \geq \begin{cases} \frac{\gamma + 4\delta}{4(\gamma+\delta)}, s \leq t \\ \frac{\alpha + 4\beta}{4(\alpha+\beta)}, t \leq s \end{cases}$$

So $\frac{G(t,s)}{G(s,s)} \geq M, \frac{1}{4} \leq t \leq 3/4$

Hence, if $u \in k$

$$\min_{\frac{1}{4} \leq t \leq 3/4} Ru(t) = \min_{\frac{1}{4} \leq t \leq 3/4} \int_0^1 G(t,s)r(s)f\big(u(s)\big)ds$$

$$\geq M\int_0^1 G(s,s)r(s)f(u(s))ds$$

$$\geq M\|Ru\|$$

Therefore,$RK \subset K$. Moverover ,it is easy to see that $R: K \to K$ is completely continuous .

Now, since $f_0 = 0$, we may choose $A_1 > 0$ so that $f(u) \leq \eta u, for\ 0 < u \leq A_1$, where $\eta > 0$ satisfies $\eta\int_0^1 G(s,s)\, r(s)ds \leq 1$ .... (2.9)

Thus if $u \in K$ and $\|u\| = A_{1,}$ then from (2.7) and (2.9)

$Ru(t) \leq \int_0^1 G(s,s)\, r(s)f\big(u(s)\big)ds \leq \|u\|, 0 \leq t \leq 1$ .... (2.10)

Now, if we consider $\Omega_1 \coloneqq \{u \in E: \|u\| < A_1\}$............................ (2.11)

Then now (2.10) shows that $\|Ru\| \leq \|u\|, u \in Kn\partial \cap_1$. Further, since $f_\infty = \infty$, there exists $A_2 > 0$ such that $f(u) \geq \mu u,\ u \geq A_2$ where $\mu > 0$ is chosen so that $M\mu\int_{1/4}^{3/4} K(1/2\ , s)r(s)ds \geq 1$........... (2.13)

Let consider $A_3 \coloneqq Max\left\{2A_1, {A_2}/{M}\right\}$ and $\Omega_2 \coloneqq \{u \in E: \|u\| < A_2\}$.Then $u \in K$ and $\|u\| = A_2$ implies $\min_{1/4<t<3/4} u(t) \geq M\|u\| > A_2$ and so

$$Ru(1/2) = \int_0^1 G(1/2, s)\, r(s)f\big(u(s)\big)d \geq \int_{1/4}^{3/4} G(1/2, s)\, r(s)f\big(u(s)\big)ds$$

Hence,$\|Ru\| \geq \|u\|$ for $u \in Kn\partial\Omega_2$

Therefore, by the first part of the fixed point theorem, it follows that R has a fixed point in $Kn\overline{\Omega_2}$ $/\overline{\Omega_1}$ such that $A_1 \leq \|u\| \leq A_3$.Further, since $G(t,s) > 0$, it follows that $u(t) > 0$ for $0 < t < 1$.

Therefore, this completes the super linear part of the theorem.

<u>Sublinear case</u>: Suppose next that $f_0 = \infty$ and $f_\infty = 0$.

We first choose $A_1 > 0$ such that $f(u) \geq \mu M$ for $0 < u \leq A_1$ where

$\mu M\int_{1/4}^{3/4} G(1/2, s)r(s)ds \geq 1$ .... (2.14) where M is as in the first part of the proof.

Then for $u \in K$ and $\|u\| = A_1$ we have $Ru(1/2) = \int_0^1 G(1/2, s)\, r(s) f(u(s)) ds \geq \int_{1/4}^{3/4} G(1/2, s)\, r(s) f(u(s)) ds$

$$\geq \mu \int_{1/4}^{3/4} G(1/2, s)\, r(s) u(s) ds$$

$$\geq \mu M \|u\| \int_{1/4}^{3/4} G(1/2, s)\, r(s) ds \geq \|u\| \text{ , by (2.14)}$$

Thus ,we may let $\Omega_1 := \{u \epsilon E : \|u\| < A_1\}$ so that $\|Ru\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$.

Now, since $f_\infty = 0$,there exists $A_3 > 0$ so that $f(u) \leq \lambda u$ for $u \geq A_3$ where $\lambda > 0$ satisfies

$$\lambda \int_0^1 G(s, s)\, r(s) ds \leq 1 \quad \text{...................... (2.15)}$$

We consider two cases.

Case 1: Suppose f is bounded, say $f(u) \leq N$, for all $u \in (0, \infty)$

In this case choose $A_2 := Max\left\{2A_1, N \int_0^1 G(s, s)\, r(s) ds\right\}$ so that for $u \in K$ with $\|u\| = A_2$.We have $Ru(t) = \int_0^1 G(t, s) r(s) f(u(s))\, ds \leq N \int_0^1 G(s, s) r(s) f(u(s))\, ds \leq A_2$

and therefore $\|Ru\| \leq \|u\|$.

Case 2:If f is unbounded ,then let $A_3 > max\{2A_1, A_2\}$ and such that

$f(u) \leq f(A_2)$,for $0 < u \leq A_2$ since f is bounded .

Then for $u \in K$ and $\|u\| = A_2$,we have $Ru(t) = \int_0^1 G(t, s)\, r(s) f(u(s)) ds$

$$\leq \int_0^1 G(s, s)\, r(s) f(u(s)) ds$$

$$\leq \int_0^1 G(s, s)\, r(s) f(A_2) ds$$

$$\leq \lambda A_2 \int_0^1 G(s, s)\, r(s) ds \leq A_2 = \|u\|$$

Therefore, in either case we may put;

$\Omega_2 := \{u \epsilon E : \|u\| < A_2\}$, and for $u \in K \cap \partial\Omega_2$, we have $\|Ru\| \leq \|u\|$. By the second part of the fixed point theorem it follows that boundary value problem (BVP) (1.1) and (1.2) has positive solution and this completes the proof of the theorem .

4. Conclusion

Therefore our conclusion is that the fixed point theorem is a basic criterion for the existence of positive solution of second order ordinary differential equation in Banach space .The main result of this paper is to investigate at least one positive solution in the case of sup linear and sub linear cases. Hence in both cases, existence of positive solution for second order ordinary differential equation has been granted.

References

[1]. A.Datta and J. Henderson, Differences and smoothness of solutions for functional difference equations, Proceedings Difference Equations 1 (1995), 133-142.

[2] S. Cheng and G. Zhang, Existence of positive periodic solutions for non-autonomous functional differential equations, Electronic Journal of Differential Equations 59 (2001), 1-8.

[3] J. Henderson and A. Peterson, Properties of delay variation in solutions of delay difference equations, Journal of Differential Equations 1 (1995), 29-38.

[4] R.P. Agarwal and P.J.Y. Wong, On the existence of positive solutions of higher order difference equations, Topological Methods in Nonlinear Analysis 10 (1997) 2, 339-351.

[5] P.W. Eloe, Y. Ra_oul, D. Reid and K. Yin, Positive solutions of nonlinear Functional Difference Equations, Computers and Mathematics With applications 42 (2001) , 639-646.

[6] C.P. Gupta and S. I. Tro_mchuk, A sharper condition for the stability of a three-point second order boundary value problem, Journal of Mathematical Analysis and Applications, 205 586-597 (1997).

[7] C.P. Gupta, Solvability of a three-point boundary value problem for a second order ordinary differential equation, Journal of Mathematical Analysis and Applications, 168 540- 551 (1997). EJQTDE, 2002 No. 15, p. 10

[8] J. Henderson and H. Wang, Positive solutions for nonlinear eigenvalue problems, Journal of Mathematical Analysis and Applications, 208 252-259 (1997).

[9] J. Henderson and W. N. Hudson, Eigenvalue problems for nonlinear di_erential equations, Communications on Applied Nonlinear Analysis 3 (1996), 51-58.

[10] M. A. Krasnosel'skii, Positive solutions of operator Equations Noordho, Groningen, (1964).

[11] R. MA, Positive solutions of a nonlinear three-point boundary-value problem , Electronic Journal of Differential Equations, Vol. 1999(1999), No.34, pp.1-8.

[12] R. MA, A remark on a second order three-point boundary value problem , Journal of Mathematical Analysis and Applications, 183, 518-522 (1994).

[13] R. MA, Existence theorem for a second order three-point boundary value problem, Journal of Mathematical Analysis and Applications, 212, 430-442 (1997).

[14] R. MA, Positive solutions for second order three-point boundary value problem, Appl.Math.Lett., 14, 193-204 (2001).

[15] F. Merdivenci, Two positive solutions of a boundary value problem for difference , Journal of Difference Equations and Application 1 (1995),

[16] W. Yin, Eigenvalue problems for functional di_erential equations, Journal of Nonlinear Differential Equations 3 (1997), 74-82.EJ

[17]. Jean.Mawhin. Michel Willem ,Critical Point Theory and Hamiltonian Systems ,Springer-Verlag New York Berlin Heidelberg ,London Paris Tokyo

[18]. L.C. Evans, Partial Differential Equations, Graduate Studies in Math.19,AMS,1997.

[19]. M.Renardy and R.C.Rogers,2nd edition,An introduction to Partial Differential Equations,2004,Springer-Verlag,New York

[20]. P. Amster. Nonlinearities in a second order ODE. Electron. J. Differ. Equ.,Pages13-21,2001. Conf. 06.

[21]. P. Amster and M.C. Mariani. A second order ODE with a nonlinear final condition. Electron. J. Differ. Equ., (75):9, 2001.

[22]. H. Brezis. Functional analysis, Sobolev spaces and partial di_erential equations. Springer,New York, 2010.

[23]. U. Ledzewicz, H. Schattler, S. Walczak, Optimal control systems governed by second-order ODEs with Dirichlet boundary data and variable Parameters, Ill. J.Math.47 (2003), No.4, 1189-1206.