# A REMARKABLE OBSERVATION ON NUMBER THEORETIC FUNCTION $\emptyset(n)$

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**ABSTRACT**

*The aim of this paper is observe the some remarkable results on number theoretic function* $\phi(n)$ *for all positive integers* $n \geq 1$. *If* $n$ *is any finite positive integer, then* $\emptyset^k(n) = 1$ *for all positive integers* $k \geq N$, *where* $1 \leq N \leq k$.

**Key Words**: Number Theoretic Function, Prime Numbers.

**Introduction**: One of the most important functions in number theory is the Euler function $\emptyset(n)$ first introduced by Euler. Euler (1707 – 1783), is universally considered as one of the greatest mathematicians in history and is certainly the most prolific among them. His researches covered almost all the fields of mathematics of his time: algebra astronomy, calculus, calculus of variations, finite differences, mechanics, number theory and several other topics. An idea of the huge output of mathematical results he discovered will be gained if one considers the fact that it required 60 to 80 large volumes to publish them. His mathematical genius was so penetrating that he discovered great principles in trivial problems. As an example of this he solved the Konigsberg riddle and thereby laid the foundation of that great branch of mathematics called Topology. During the last seventeen years of his life totally blind, but this calamity did not prevent him from continuing his researches with greater vigor and producing results of first importance {[2.2.1] S. G. Telang}.

**Definition (1)**: Let $m$, $n$ are two given positive integers. Then $m, and\ n$ is said to be relatively prime numbers if $gcd(m\ , n) = 1$.

**Definition (2)**: Let $n$ be a given positive integer. Then $\emptyset(\boldsymbol{n})$ is the number of relatively prime numbers to $n$ which not exceeding $n$.

**Examples**:

(1) Let $n = 1$. Hence $\emptyset(\boldsymbol{n}) = 1$.

(2) The relatively prime numbers to 12 are 1, 5, 7, and 11. Hence $\emptyset(\boldsymbol{n}) = 4$.

(3) The relatively prime numbers to 7 are 1, 2, 3, 4, 5, and 6. Hence $\emptyset(\boldsymbol{n}) = 6$.

**Theorem (1)**: If $n$ is a prime number, then $\emptyset(\boldsymbol{n}) = (n-1)$.

**Proof**: Since we know that every prime number has only two divisors 1 and itself. Therefore the relatively prime numbers to $n$ will be $(n-1)$. Hence $\emptyset(\boldsymbol{n}) = (n-1)$.

**Theorem (2)**: Let $n = p^{k_1}$, then $\emptyset(n) = ({p_1}^{k_1} - {p_1}^{k_1-1})$.

**Proof**: 1. Let $k_1 = 1$. Then integers less than $n$ and prime to it are 1, 2,$\ldots\ldots, p-1$. Hence $\emptyset(\boldsymbol{n}) = (p-1)$.

2. Let $k_1 > 1$. Then integers less than $n$ and prime to it are $p, 2p, 3p, \ldots\ldots. p^{k_1}, p^{k_1-1}$. Hence

$$\emptyset(n) = ({p_1}^{k_1} - {p_1}^{k_1-1}).$$

**Theorem (3)**: Let $n = {p_1}^{k_1} {p_2}^{k_2} {p_3}^{k_3} \ldots\ldots\ldots\ldots {p_r}^{k_r}$, then

$$\emptyset(n) = ({p_1}^{k_1} - {p_1}^{k_1-1})({p_2}^{k_2} - {p_2}^{k_2-1}) \ldots ({p_r}^{k_r} - {p_r}^{k_r-1})$$

**Proof**: Since $gcd(p_i, p_j) = 1$, for $\forall i \neq j$.

$\emptyset(n)$ = number of integers $\leq n$ which are prime to $n$.

= the number of integers $\leq n$ which are not divisible by $p_1, p_2, p_3, \ldots\ldots\ldots\ldots, p_r$

$= ({p_1}^{k_1} - {p_1}^{k_1-1})({p_2}^{k_2} - {p_2}^{k_2-1}) \ldots ({p_r}^{k_r} - {p_r}^{k_r-1})$

**Theorem (4)**: Let $m$ and $n$ are two relatively primes, then $\emptyset(mn) = \emptyset(m)\emptyset(n)$.

**Proof**: Let $m = p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots\ldots\ldots p_r^{k_r}$ and $n = q_1^{j_1} q_2^{j_2} q_3^{j_3} \ldots\ldots\ldots q_s^{j_s}$ be the canonical representation of $m$ and $n$. Since $gcd(m, n) = 1$ the $p$ primes above are all different from the $q$ primes. Hence

$$\emptyset(mn) = \emptyset(p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots\ldots\ldots . p_r^{k_r} \times q_1^{j_1} q_2^{j_2} q_3^{j_3} \ldots\ldots\ldots q_s^{j_s})$$

$$= (p_1^{k_1} - p_1^{k_1 - 1})(p_2^{k_2} - p_2^{k_2 - 1}) \ldots (p_r^{k_r} - p_r^{k_r - 1}) \times (q_1^{j_1} - q_1^{j_1 - 1})(q_2^{j_2} - q_2^{j_2 - 1}) \ldots (q_s^{j_s} - q_s^{j_s - 1})$$

$$= \emptyset(m)\emptyset(n).$$

**Corollary (1)**: Let $n = p\,q$, where $p$ and $q$ are two prime numbers such that $p \neq q$.

Hence $\emptyset(pq) = pq - p - q + 1.$

**Proof:** Since we know that $p$ and $q$ are two prime numbers such that $p \neq q$ therefore $gcd(p, q) = 1$. By using theorem (1) & (4) we get $\emptyset(pq) = \emptyset(p)\emptyset(q) = (p-1)(q-1) = pq - p - q + 1.$

Hence $\emptyset(pq) = pq - p - q + 1.$

**Result (1)**: Let $n_1 = p^2$, then $\emptyset(n_1) = (p^2 - p) = p(p-1).$

**Proof**: Using theorem (2), we get required result. Similarly we can find other results which are given below.

**Result (2)**: Let $n_2 = p^2 q$, where $p \neq q$ then

$$\emptyset(n_2) = (p^2 - p).(q-1) = p\,(p-1)(q-1).$$

**Result (3)**: Let $n_3 = p^2 q^2$, where $p \neq q$ then

$$\emptyset(n_3) = (p^2 - p).(q^2 - q) = p\,(p-1)q(q-1).$$

$$= p\,q\,(p-1)(q-1).$$

**Result (4)**: Let $n_4 = p^3 q$, where $p \neq q$ then

$$\emptyset(n_4) = (p^3 - p^2).(q - 1) = p^2 (p-1)(q-1)$$

**Result (5)**: Let $n_5 = p^3 q^2$ , where $p \neq q$ then

$$\emptyset(n_5) = (p^3 - p^2).(q^2 - q) = p^2 (p-1)q (q-1).$$

$$= p^2 q (p-1)(q-1).$$

**Result (6)**: Let $n_6 = p^3 q^3$ , where $p \neq q$ then

$$\emptyset(n_6) = (p^3 - p^2).(q^3 - q^2) = p^2 (p-1)q^2 (q-1).$$

$$= p^2 q^2 (p-1)(q-1).$$

**Result (7)**: Let $n_7 = p^3$ , where $p \neq q$ then

$$\emptyset(n_7) = (p^3 - p^2) = p^2 (p-1).$$

**Result (8)**: Let $n_8 = p^4$ , where $p \neq q$ then

$$\emptyset(n_8) = (p^4 - p^3) = p^3 (p-1).$$

**Result (9)**: Let $n_9 = p^4 q$ , where $p \neq q$ then

$$\emptyset(n_9) = (p^4 - p^3).(q - 1) = p^3 (p-1)(q-1).$$

**Result (10)**: Let $n_{10} = p^4 q^2$ , where $p \neq q$ then

$$\emptyset(n_{10}) = (p^4 - p^3).(q^2 - q) = p^3 (p-1)q (q-1).$$

$$= p^3 q (p-1)(q-1).$$

**Result (11)**: Let $n_{11} = p^5$ , then

$$\emptyset(n_{11}) = (p^5 - p^4) = p^4 (p-1).$$

**Result (12)**: Let $n_{12} = p^6$ , then

$$\emptyset(n_{12}) = (p^3 - p^2).(q^3 - q^2) = p^2 (p-1)q^2 (q-1) = p^2 q^2 (p-1)(q-1).$$

**Result (13)**: Let $n_{13} = p^5 q$ , where $p \neq q$ then

$$\emptyset(n_{13}) = (p^5 - p^4).(q-1) = p^4\,(p-1)(q-1).$$

**Definition (2):** Let $\emptyset(\boldsymbol{n})$ is the number of relatively prime numbers to $n$ which not exceeding $n$, then $\emptyset^k(n)$ is also exist for all positive integers $k \geq 1$ .

**Verification of the above definition (2):**

For the verification of the above definition we construct a table (given below) for positive integers $n$, where $1 \leq n \leq 100$.

| $n$ | $\phi(\mathrm{n})$ | $\emptyset^2(n)$ | $\emptyset^3(n)$ | $\emptyset^4(n)$ | $\emptyset^5(n)$ | $\emptyset^6(n)$ | $\emptyset^7(n)$ | $\ldots\ldots\ldots..\emptyset^k(n)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 4 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 6 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 4 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 6 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 4 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 10 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 14 | 6 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 15 | 8 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| $n$ | $\phi(n)$ | $\emptyset^2(n)$ | $\emptyset^3(n)$ | $\emptyset^4(n)$ | $\emptyset^5(n)$ | $\emptyset^6(n)$ | $\emptyset^7(n)$ | $\ldots\ldots\ldots..\emptyset^k(n)$ |
| 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 17 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 18 | 6 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 19 | 18 | 6 | 2 | 1 | 1 | 1 | 1 | 1 |
| 20 | 8 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 21 | 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 22 | 10 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 23 | 22 | 10 | 4 | 2 | 1 | 1 | 1 | 1 |
| 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 25 | 20 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 26 | 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 27 | 18 | 6 | 2 | 1 | 1 | 1 | 1 | 1 |
| 28 | 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 29 | 28 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 30 | 8 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 31 | 30 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 32 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 33 | 20 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 34 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 35 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 36 | 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 37 | 36 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 38 | 18 | 6 | 2 | 1 | 1 | 1 | 1 | 1 |
| 39 | 38 | 18 | 6 | 2 | 1 | 1 | 1 | 1 |
| 40 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 41 | 40 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 42 | 12 | 4 | 2 | 1 | 1 | 1 | 1 | 1 |
| 43 | 42 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 44 | 20 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 45 | 30 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 46 | 22 | 10 | 4 | 2 | 1 | 1 | 1 | 1 |
| $n$ | $\phi(n)$ | $\emptyset^2(n)$ | $\emptyset^3(n)$ | $\emptyset^4(n)$ | $\emptyset^5(n)$ | $\emptyset^6(n)$ | $\emptyset^7(n)$ | $\ldots\ldots\ldots..\emptyset^k(n)$ |
| 47 | 46 | 22 | 10 | 4 | 2 | 1 | 1 | 1 |
| 48 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 49 | 42 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 50 | 20 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 51 | 32 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 52 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 53 | 52 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 54 | 18 | 6 | 2 | 1 | 1 | 1 | 1 | 1 |
| 55 | 40 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 56 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 57 | 36 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 58 | 28 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 59 | 58 | 28 | 12 | 4 | 2 | 1 | 1 | 1 |
| 60 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 61 | 60 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 62 | 30 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 63 | 36 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 64 | 32 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 65 | 48 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 66 | 20 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 67 | 66 | 20 | 8 | 4 | 2 | 1 | 1 | 1 |
| 68 | 32 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 69 | 44 | 20 | 8 | 4 | 2 | 1 | 1 | 1 |
| 70 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 71 | 70 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 72 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 73 | 72 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 74 | 36 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 75 | 50 | 20 | 8 | 4 | 2 | 1 | 1 | 1 |
| 76 | 36 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 77 | 60 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| $n$ | $\phi(n)$ | $\emptyset^2(n)$ | $\emptyset^3(n)$ | $\emptyset^4(n)$ | $\emptyset^5(n)$ | $\emptyset^6(n)$ | $\emptyset^7(n)$ | $\dots\dots\dots..\emptyset^k(n)$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 78 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 79 | 78 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 80 | 32 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 81 | 54 | 18 | 6 | 2 | 1 | 1 | 1 | 1 |
| 82 | 40 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 83 | 82 | 40 | 16 | 8 | 4 | 2 | 1 | 1 |
| 84 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 85 | 64 | 32 | 16 | 8 | 4 | 2 | 1 | 1 |
| 86 | 42 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 87 | 56 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 88 | 40 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 89 | 88 | 40 | 16 | 8 | 4 | 2 | 1 | 1 |
| 90 | 24 | 8 | 4 | 2 | 1 | 1 | 1 | 1 |
| 91 | 72 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 92 | 44 | 20 | 8 | 4 | 2 | 1 | 1 | 1 |
| 93 | 60 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 94 | 46 | 22 | 10 | 4 | 2 | 1 | 1 | 1 |
| 95 | 72 | 24 | 8 | 4 | 2 | 1 | 1 | 1 |
| 96 | 32 | 16 | 8 | 4 | 2 | 1 | 1 | 1 |
| 97 | 96 | 32 | 16 | 8 | 4 | 2 | 1 | 1 |
| 98 | 42 | 12 | 4 | 2 | 1 | 1 | 1 | 1 |
| 99 | 60 | 4 | 16 | 8 | 4 | 2 | 1 | 1 |
| 100 | 40 | 3 | 16 | 8 | 4 | 2 | 1 | 1 |

From the above table we find the following results:

**Result (13):** If $n = 1 \& 2$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 1$ .

**Result (14):** If $n = 3, 4, 6$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 2$ .

**Result (15):** If $n = 5, 7, 8 \ldots .10, 12, 14, 18$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 3$ .

**Result (16):** If $n = 11, 13, 15, 16\ etc.$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 4$ .

**Result (17):** If $n = 17, 23, 25, 29\ etc.$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 5$ .

**Result (18):** If $n = 41, 47, 51, 53\ etc.$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 6$ .

**Result (19):** If $n = 83, 85, 89, 97\ etc.$, then $\emptyset^k(n) = 1$, for all positive integers $k \geq 7$ .

**Conclusions:**

If $n$ is any finite positive integer, we find out the following result $\emptyset^k(n) = 1$ for all positive integers $k \geq N$, where $1 \leq N \leq k$. Also we can say that $\lim_{k\to\infty} \emptyset^k(n) = 1$.

**References**

1. David M. Burton, Elementary Number Theory, Tata McGraw Hill Private Limited, New Delhi, 2008.
2. Gareth A. Jones and J. Mary Jones, Elementary Number Theory, Springer-Verlag London, 2006.
3. Ivan Niven, Herbert S. Zuckerman and Hugh L. Montgomery, An Introduction To The Theory Of Numbers , John Wiley & Sons, Inc., 2004.
4. Kenneth Ireland Michael Rosen, A Classical Introduction to Modern Number Theory, Springer-Verlag New York, 2005.
5. S. G. Telang; Tata McGraw- Hill Publishing Company Limited; New Delhi.