GENERALIZATION OF THE JACOBSTHAL SEQUENCE

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ABSTRACT

In this note I have explored a new generalization of Jacobsthal sequence using some arbitrary real numbers and derived relation among them.

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Key Word: Jacobsthal sequence,

[1] Introduction: Jacobsthal Sequence [3] is defined as

$$J_0 = 0 \quad J_1 = 1 \qquad J_n = J_{n-1} + 2J_{n-2} \qquad n \geq 2 \tag{1}$$

Let us define two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ where *a, b, c* and *d* are arbitrary real numbers such that

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \alpha_{n+2} = \beta_{n+1} + 2\beta_n, \quad n \geq 0 \\ \beta_{n+2} = \alpha_{n+1} + 2\alpha_n, \quad n \geq 0 \end{cases} \tag{2}$$

On setting *a=b* and *c=d*, the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other. In particular on setting *a=b=*0 and *c=d=*1 we get the Jacobsthal sequence and setting *a=b=2* and *c=d=1* we get the Jacobsthal -Lucas sequence. The first ten terms of the sequences defined above are :

n	α_n	β_n
0	a	b
1	c	d
2	2b+d	2a+c

3	2a+c+2d	2b+2c+d
4	4a+2b+4c+d	2a+4b+c+4d
5	2a+8b+5c+6d	8a+2b+6c+5d
6	12a+10b+8c+13d	10a+12b+13c+8d
7	26a+16b+25c+18d	16a+26b+18c+25d
8	36a+50b+44c+41d	50a+36b+41c+44d
9	82a+88b+77c+94d	88a+82b+94c+77d

If we express the members of the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$, when $n \geq 0$ as

$$\begin{cases} \alpha_n = \Gamma_n^1 a + \Gamma_n^2 b + \Gamma_n^3 c + \Gamma_n^4 d \\ \beta_n = \delta_n^1 a + \delta_n^2 b + \delta_n^3 c + \delta_n^4 d \end{cases} \quad (3)$$

We obtain the eight sequences $\{\Gamma_i^j\}_{i=0}^{\infty}$ and $\{\delta_i^j\}_{i=0}^{\infty}$, (j = 1, 2, 3, 4). These eight sequences are related to each other and to the Jacobsthal numbers. These relations are shown here in the form of theorems.

[2] Theorems On Related Sequences

Theorem 1 :

(a) $\Gamma_n^1 + \delta_n^1 = J_{n-1}, \quad n > 0$ **(c)** $\Gamma_n^3 + \delta_n^3 = J_n, \quad n \geq 0$

(b) $\Gamma_n^2 + \delta_n^2 = 2J_{n-1}, \quad n > 0$ **(d)** $\Gamma_n^4 + \delta_n^4 = J_n, \quad n \geq 0$

Proof: **(a)** This is obviously true if n = **0** and **1**, since $\Gamma_1^1 + \delta_1^1 = 0 + 0 = 0 = J_0$

Assume this statement be true for $n \geq 1$. Then

$$\Gamma_{n+1}^1 + \delta_{n+1}^1 = \delta_n^1 + 2\delta_{n-1}^1 + \Gamma_n^1 + 2\Gamma_{n-1}^1 \quad \text{By (2)}$$

$$= \left(\delta_n^1 + \Gamma_n^1\right) + 2\left(\delta_{n-1}^1 + \Gamma_{n-1}^1\right)$$

$= J_{n-1} + 2J_{n-2}$ (By induction hypothesis)

$= J_n$ (By definition of Jacobsthal number)

Hence (a) is true for all $n > 0$ by mathematical induction. Similar proofs can be given for parts (b), (c) and (d).

Theorem 2 : If $n \geq 0$, then

(a) $\Gamma_n^1 + \Gamma_n^2 = \delta_n^1 + \delta_n^2$

(b) $\Gamma_n^3 + \Gamma_n^4 = \delta_n^3 + \delta_n^4$

(c) $\Gamma_n^1 + \Gamma_n^2 + \Gamma_n^3 + \Gamma_n^4 = \delta_n^1 + \delta_n^2 + \delta_n^3 + \delta_n^4$

Proof: (a) Obviously this is true for $n = 0$ and 1. Let it be true for some integer $n \geq 2$. Then,

$\Gamma_{n+1}^1 + \Gamma_{n+1}^2 = \delta_n^1 + 2\delta_{n-1}^1 + \delta_n^2 + 2\delta_{n-1}^2$ By (2)

$= \left(\delta_n^1 + \delta_n^2\right) + 2\left(\delta_{n-1}^1 + \delta_{n-1}^2\right)$

$= \left(\Gamma_n^1 + \Gamma_n^2\right) + 2\left(\Gamma_{n-1}^1 + \Gamma_{n-1}^2\right)$ (By induction hypothesis)

$= \left(\Gamma_n^1 + 2\Gamma_{n-1}^1\right) + \left(\Gamma_n^2 + 2\Gamma_{n-1}^2\right)$

$= \delta_{n+1}^1 + \delta_{n+1}^2$ By (2)

Hence, by mathematical induction (a) is true for all $n \geq 0$. Similarly we may have (b). Part (c) may have by addition of (a) and (b).

Theorem 3: If $n \geq 0$, then

(a) $\delta_n^1 = \Gamma_n^2$ **(b)** $\delta_n^2 = \Gamma_n^1$

(c) $\delta_n^3 = \Gamma_n^4$ **(d)** $\delta_n^4 = \Gamma_n^3$

(e) $2\Gamma_n^3 = \Gamma_{n+1}^2$ **(f)** $2\Gamma_n^4 = \Gamma_{n+1}^1$

(g) $2\delta_n^3 = \delta_{n+1}^2$ **(h)** $2\delta_n^4 = \delta_{n+1}^1$

Proof: (a) The statement is true for $n = 0, 1, 2$. Let be true for integer $n \geq 3$. Then,

$$\delta_{n+1}^1 = \Gamma_n^1 + 2\Gamma_{n-1}^1 \quad \text{By (2)}$$

$$= \left(\delta_{n-1}^1 + 2\delta_{n-2}^1\right) + 2\left(\delta_{n-2}^1 + 2\delta_{n-3}^1\right) \quad \text{By (2)}$$

$$= \left(\Gamma_{n-1}^2 + 2\Gamma_{n-2}^2\right) + 2\left(\Gamma_{n-2}^2 + 2\Gamma_{n-3}^2\right) \quad \text{(By induction hypothesis)}$$

$$= \delta_n^2 + 2\delta_{n-1}^2 \quad \text{By (2)}$$

$$= \Gamma_{n+1}^2 \quad \text{By (2)}$$

Hence, (a) is true for all $n \geq 0$ by mathematical induction. Similarly we can have other proofs.

Theorem 4: **(a)** $\Gamma_n^1 + \Gamma_n^2 = \delta_n^1 + \delta_n^2 = 2J_n \quad (n > 0)$

(b) $\Gamma_n^3 + \Gamma_n^4 = \delta_n^3 + \delta_n^4 = J_n \quad (n > 0)$

Using theorem (2) we may follow the result.

Theorem 5: **(a)** $\sum_{i=1}^{k} \Gamma_i^1 = \sum_{i=1}^{k} \delta_i^2$ **(b)** $\sum_{i=1}^{k} \Gamma_i^2 = \sum_{i=1}^{k} \delta_i^1$

(c) $\sum_{i=1}^{k} \Gamma_i^3 = \sum_{i=1}^{k} \delta_i^4$ **(d)** $\sum_{i=1}^{k} \Gamma_i^4 = \sum_{i=1}^{k} \delta_i^3$

Proof: (a) Since by Theorem (3) $\delta_i^2 = \Gamma_i^1 \quad \forall\, i = 1,2,3 \ldots\ldots$

$$\therefore \quad \sum_{i=1}^{k} \Gamma_i^1 = \sum_{i=1}^{k} \delta_i^2 .$$

Similar Proofs can be given for other parts.

Theorem 6: $\alpha_{n+2} + \beta_{n+2} = 2J_{n+1}(\alpha_0 + \beta_0) + J_{n+2}(\alpha_1 + \beta_1) \quad (n \geq 0)$

Proof: The statement is true for $n = 0$ and 1. Assume this be true for some integer $n \geq 2$. Then,

$$\alpha_{n+3} + \beta_{n+3} = (\beta_{n+2} + 2\beta_{n+1}) + (\alpha_{n+2} + 2\alpha_{n+1}) \qquad \text{By (2)}$$

$$= (\alpha_{n+2} + \beta_{n+2}) + 2(\alpha_{n+1} + \beta_{n+1})$$

$$= 2J_{n+1}(\alpha_0 + \beta_0) + J_{n+2}(\alpha_1 + \beta_1) + 2\{2J_n(\alpha_0 + \beta_0) + J_{n+1}(\alpha_1 + \beta_1)\}$$

(By induction hypothesis)

$$= 2(J_{n+1} + 2J_n)(\alpha_0 + \beta_0) + (J_{n+2} + 2J_{n+1})(\alpha_1 + \beta_1)$$

$$= J_{n+2}(\alpha_0 + \beta_0) + 2J_{n+1}(\alpha_1 + \beta_1)$$

(By definition of Jacobsthal number)

Hence, by mathematical induction the statement is true for all integer $n \geq 0$.

Theorem 7: If $n \geq 2$, then

(a) $\Gamma_n^1 = \Gamma_{n-1}^2 + 2\Gamma_{n-2}^2$

(b) $\Gamma_n^2 = \Gamma_{n-1}^1 + 2\Gamma_{n-2}^1$

(c) $\Gamma_n^3 = \Gamma_{n-1}^4 + 2\Gamma_{n-2}^4$

(d) $\Gamma_n^4 = \Gamma_{n-1}^3 + 2\Gamma_{n-2}^3$

(e) $\delta_n^1 = \delta_{n-1}^2 + 2\delta_{n-2}^2$

(f) $\delta_n^2 = \delta_{n-1}^1 + 2\delta_{n-2}^1$

(g) $\delta_n^3 = \delta_{n-1}^4 + 2\delta_{n-2}^4$

(h) $\delta_n^4 = \delta_{n-1}^3 + 2\delta_{n-2}^3$

Proof: (a) The statement is true for $n = 2$ and 3. Let it be true for all integer $n \geq 4$. Then,

$$\Gamma^{1}_{n+1} = \delta^{1}_{n} + 2\delta^{1}_{n-1} \quad \text{By (2)}$$

$$= \Gamma^{2}_{n} + 2\Gamma^{2}_{n-1} \quad \text{By Theorem (3)}$$

Hence, (a) is true for all $n \geq 0$ by mathematical induction. Similar proofs can be given for other parts.

[3] Further Scope:

The sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ can also be expressed as follows.

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \alpha_{n+2} = \alpha_{n+1} + 2\alpha_n, \quad n \geq 0 \\ \beta_{n+2} = \beta_{n+1} + 2\beta_n, \quad n \geq 0 \end{cases} \tag{3}$$

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \alpha_{n+2} = \beta_{n+1} + 2\alpha_n, \quad n \geq 0 \\ \beta_{n+2} = \alpha_{n+1} + 2\beta_n, \quad n \geq 0 \end{cases} \tag{4}$$

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \alpha_{n+2} = \alpha_{n+1} + 2\beta_n, \quad n \geq 0 \\ \beta_{n+2} = \beta_{n+1} + 2\alpha_n, \quad n \geq 0 \end{cases} \tag{5}$$

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