APPELL'S AND HUMBERT'S DISTRIBUTIONS OF MATRIX VARIATE IN THE COMPLEX CASE

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ABSTRACT

The aim of this paper is to investigate matrix variate generalizations of multivariate Appell's and Humbert's families of distributions in the complex case. The multivatiate Appell's and Humbert's families of distributions have been proposed and studied recently by Mathai and also by Saxana,Sethi and Gupta. Many known or new results have been made with the help of Appell's functions of matrix arguments-one each for the function $\bar{F}_1$ and $\bar{F}_3$ and one for the function $\bar{F}_2$ in complex case.

INTRODUCTION

Appell's functions of matrix arguments have earlier been studied by Mathai [4, 5, 6] and also by Saxena, Sethi and Gupta [7]. In the present paper we have utilized Mathai's definitions for all the functions studied. Upadhyaya and Dhani have given integral representation associated with Appell's Humbert's functions of matrix arguments in the case of real symmetric positive definite. We have given in this paper further generalization of these results in the case of Hermitian positive definite matrix of complex number. All the matrices appearing in this in the paper are (pxp) real Hermitian positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [3, 4].

1. Preliminary Definitions

Definition 1.1: The Appell's function $\bar{F}_1 = \bar{F}_1$ (a, b, b′; c; – $\bar{X} - \bar{Y}$) of matrix argument is defined as that function for which the M-transform (Matrix-transform) is the following:

$$M(\bar{F}_1) = \int_{\bar{X}>0}\int_{\bar{Y}>0} |\bar{X}|^{\rho_1 - p}\, |Y|^{\rho_2 - p} F_1(a, b, b'; c; -\bar{X} - \bar{Y})\, d\bar{X}\, d\bar{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1-\rho_2)\tilde{\Gamma}_p(b-\rho_1)\tilde{\Gamma}_p(b-\rho_2)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(b')\tilde{\Gamma}_p(c-\rho_1-\rho_2)} \quad \ldots. \quad (1.1)$$

for Re ($a - \rho_1 - \rho_2$, $b - \rho_1$, $b' - \rho_2$, $c - \rho_1 - \rho_2$, ρ_1, ρ_2) > p-1

Definition 1.2: $\bar{F}_2 = \bar{F}_2$ (a, b, b′; c; – $\bar{X} - \bar{Y}$)

$$M(\bar{F}_2) = \int_{\bar{X}>0}\int_{\bar{Y}>0} |\bar{X}|^{\rho_1 - p}\, |Y|^{\rho_2 - p} F_2(a, b, b'; c, c; -\bar{X} - \bar{Y})\, d\bar{X}\, d\bar{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(c')\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1-\rho_2)\tilde{\Gamma}_p(b-\rho_1)\tilde{\Gamma}_p(b'-\rho_2)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(b')\tilde{\Gamma}_p(c-\rho_1)\tilde{\Gamma}_p(c-\rho_2)} \quad \ldots. \quad (1.2)$$

for Re ($a - \rho_1 - \rho_2$, $b - \rho_1$, $b' - \rho_2$, $c - \rho_1$, $c' - \rho_2$, ρ_1, ρ_2) > p-1

Definition 1.3: $\bar{F}_3 = \bar{F}_3$ (a, a′, b, b′; c; – $\bar{X} - \bar{Y}$)

$$M(\bar{F}_3) = \int_{\bar{X}>0}\int_{\bar{Y}>0} |\bar{X}|^{\rho_1 - p}\, |Y|^{\rho_2 - p} F_3(a, a'\, b, b'; c; -\bar{X} - \bar{Y})\, d\bar{X}\, d\bar{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1(a'-\rho_2)\tilde{\Gamma}_p(b-\rho_1)\tilde{\Gamma}_p(b'-\rho_2)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(a')\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(b')\tilde{\Gamma}_p(c-\rho_1-\rho_2)} \quad \ldots. \quad (1.3)$$

for Re ($a - \rho_1$, $a' - \rho_2$, $b - \rho_1$, $b' - \rho_2$, $c - \rho_1 - \rho_2$, ρ_1, ρ_2) > p-1

Definition 1.4: $\overline{F}_4 = \overline{F}_4$ (a, b; c, c′; – $\overline{X} - \overline{Y}$)

$$M(\overline{F}_4) = \int_{\overline{X}>0}\int_{\overline{Y}>0} |\overline{X}|^{\rho_1 - p} |Y|^{\rho_2 - p} F_4(a, b; c, c'; -\overline{X} - \overline{Y})\, d\overline{X}\, d\overline{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(c')\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1-\rho_2)\tilde{\Gamma}_p(b-\rho_1-\rho_2)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c-\rho_1)\tilde{\Gamma}_p(c'-\rho_2)} \quad \ldots. \quad (1.4)$$

for Re $(a - \rho_1 - \rho_2, b - \rho_1 - \rho_2, c - \rho_1 - \rho_2, \rho_1, \rho_2) > p-1$

Definition 1.5: The Humbert's function $\overline{\Phi}_1 = \overline{\Phi}_1$ (a, b; c; – $\overline{X} - \overline{Y}$) of matrix arguments is defined as that function which has the following M-transform:

$$M(\overline{\Phi}_1) = \int_{\overline{X}>0}\int_{\overline{Y}>0} |\overline{X}|^{\rho_1 - p} |Y|^{\rho_2 - p}\ \overline{\Phi}_1(a, b; c; -\overline{X}, -\overline{Y})\, d\overline{X}\, d\overline{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1-\rho_2)\tilde{\Gamma}_p(b-\rho_1)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c-\rho_1-\rho_2)} \quad \ldots. \quad (1.5)$$

for Re $(a - \rho_1 - \rho_2, b - \rho_1, c - \rho_1 - \rho_2, \rho_1, \rho_2) > p-1$

Definition 1.6: $\overline{\Psi}_1 = \overline{\Psi}_1$ (a, b; c; – $\overline{X} - \overline{Y}$)

$$M(\overline{\Psi}_1) = \int_{\overline{X}>0}\int_{\overline{Y}>0} |\overline{X}|^{\rho_1 - p} |Y|^{\rho_2 - p\text{-}1}\ \overline{\Psi}_1(a, b; c, c'; -\overline{X}, -\overline{Y})\, d\overline{X}\, d\overline{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(c')\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1-\rho_2)\tilde{\Gamma}_p(b-\rho_1)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c-\rho_1)\tilde{\Gamma}_p(c'-\rho_2)} \quad \ldots. \quad (1.6)$$

for Re $(a - \rho_1 - \rho_2, b - \rho_1, c - \rho_1\ c' - \rho_2, \rho_1, \rho_2) > p-1$

Definition 1.7: $\overline{\Psi}_2 = \overline{\Psi}_2$ (a; c, c′; – $\overline{X} - \overline{Y}$)

$$M(\overline{\Psi}_2) = \int_{\overline{X}>0}\int_{\overline{Y}>0} |\overline{X}|^{\rho_1 - p} |Y|^{\rho_2 - p\text{-}1}\ \overline{\Psi}_2(a, c, c'; -\overline{X}, -\overline{Y})\, d\overline{X}\, d\overline{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(c')\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1-\rho_2)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c-\rho_1)\tilde{\Gamma}_p(c'-\rho_2)} \qquad \ldots. \qquad (1.7)$$

for Re $(a - \rho_1 - \rho_2,\ c - \rho_1,\ c' - \rho_2,\ \rho_1,\ \rho_2) > p\text{-}1$

Definition 1.8: $\bar{\Xi}_2 = \bar{\Xi}_2\ (a; b; c; -\bar{X} - \bar{Y})$

$$M(\bar{\Xi}_2) = \int \bar{X} > 0 \int \bar{Y} > 0\ |\bar{X}|\rho_1^{-p}\ |Y|\rho_2^{-p\text{-}1}\ \bar{\Xi}_2(a, b; c; -\bar{X}, -\bar{Y})\ d\bar{X}\ d\bar{Y}$$

$$= \frac{\tilde{\Gamma}_p(c)\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a-\rho_1)\tilde{\Gamma}_p(b-\rho_1)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c-\rho_1-\rho_2)} \qquad \ldots. \qquad (1.8)$$

for Re $(a - \rho_1,\ b - \rho_1,\ c - \rho_2,\ \rho_1,\ \rho_2) > p\text{-}1$

2. Appell's Functions of Matrix Arguments.

THEOREM 2.1:

$\bar{F}_1\ (\alpha, \beta, \beta'; c; -\bar{X} - \bar{Y})$

$$= \frac{\tilde{\Gamma}_p(\gamma)}{\tilde{\Gamma}_p(\beta)\tilde{\Gamma}_p(\beta')\tilde{\Gamma}_p(\gamma-\beta-\beta')} \int_0^I \int_0^I |\bar{U}|^{\beta+\beta'-p}\ |\bar{V}|^{\beta'-p} \times |I-\bar{U}|^{\gamma-\beta+\beta'-p}\ |I-\bar{V}|^{\beta-p}$$

$$\left|I-(I-\bar{V})^{1/2}\bar{U}^{1/2}X\bar{U}^{1/2}(I-\bar{V})^{1/2}+\bar{V}^{1/2}\bar{U}^{1/2}Y\bar{U})^{1/2}(I-\bar{V})^{1/2}\right|^{-\alpha} d\bar{U}\ d\bar{V} \qquad \ldots (2.1)$$

for $0 < \bar{U} < I$, $0 < \bar{V} < I$ and for Re $(\beta, \beta', \gamma - \beta - \beta') > p - 1$

PROOF: We take the M-transform of the right side of eq. (2.1) with respect to the variables $\bar{X}$ and $\bar{Y}$ and the parameters ρ_1 and ρ_2 respectively to obtain,

$$\int \overline{X} > 0 \int \overline{Y} > 0 \, |\overline{X}|\rho_1^{-p} \, |\overline{Y}|\rho_2^{-p-1} \, \left| I - (I - \overline{V})^{1/2} \overline{U}^{1/2} X \overline{U}^{1/2} (I - \overline{V})^{1/2} + \overline{V}^{1/2} \overline{U}^{1/2} \overline{Y} \overline{U})^{1/2} \overline{V})^{1/2} \right|^{-\alpha} d$$

$\overline{X}$ d $\overline{Y}$ ... (2.2)

Making use of the transformations,

$\overline{X}_1 = (I - \overline{V})^{1/2} \overline{X} \, \overline{U}^{1/2}$, $\overline{Y}_1 = \overline{V}^{1/2} \overline{Y} \, \overline{U}^{1/2} \overline{V}^{1/2}$ with, $d\overline{X}_1 = |I - \overline{V}|^p |\overline{U}|^p d\overline{X}$, $d\overline{Y}_1 = |\overline{V}|^p |\overline{U}|^p d\overline{Y}$ and $|\overline{X}_1| = |I - \overline{V}| |\overline{U}| |\overline{X}|$, $|\overline{Y}_1| = |\overline{V}| |\overline{U}| |\overline{Y}|$

in the above expression and then integration out the variables $\overline{X}_1$ and $\overline{Y}_1$ by using a type – 2 Dirichlet integral we get,

$$|\overline{U}|^{-\rho_1-\rho_2} |\overline{V}|^{-\rho_2} |I - \overline{V}|^{-\rho_1} \frac{\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(a - \rho_1 - \rho_2)}{\tilde{\Gamma}_p(\alpha)} \quad \ldots (2.3)$$

Substituting this expression on the right side of eq. (2.1) and then integrating out the variables $\overline{U}$ and $\overline{V}$ in the resulting expression by using a type – 1 Beta integral generates M($\overline{F}_1$) as give by eq. (1.1)

THEOREM 2.2:

$$|\overline{P}|\beta' \; F_2(\alpha, \beta, \beta'; \gamma, \gamma'; -\overline{X}, -\overline{P}^{-1/2} \overline{Y} - \overline{P}^{-1/2})$$

$$= \frac{1}{\tilde{\Gamma}_p(\beta')} \int \overline{T} > 0 e^{-tr(\overline{P}\,\overline{T})} |\overline{T}|^{\beta' - P} \; \overline{\Psi}_1(\alpha, \beta; \gamma, \gamma'; -\overline{X} - \overline{T}^{1/2} \overline{Y} \, \overline{T}^{1/2}) \quad \ldots (2.4)$$

for Re (β′) > – 2.

PROOF: Taking the M-transform of the right side of eq. (2.4) with respect to the variables $\overline{X}$ and $\overline{Y}$ and the parameters ρ_1 and ρ_1 respectively, we get,

$$\int \overline{X} > 0 \int \overline{Y} > 0 \, |\overline{X}|\rho_1^{-p} \, |\overline{Y}|\rho_2^{-p} \; \overline{\Psi}_1(\alpha, \beta; \gamma, \gamma'; -\overline{X} - \overline{T}^{1/2} \overline{Y} \, \overline{T}^{1/2}) d\overline{X} \, d\overline{Y} \quad \ldots (2.5)$$

and then the use of definition (1.6) leads us to,

$$|\bar{T}|^{-\rho_2} \frac{\tilde{\Gamma}_p(\gamma)\tilde{\Gamma}_p(\gamma')\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\alpha-\rho_1-\rho_2)\tilde{\Gamma}_p(\beta-\rho_1)}{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)\tilde{\Gamma}_p(\gamma-\rho_1)\tilde{\Gamma}_p(\gamma'-\rho_2)} \quad \ldots (2.6)$$

Substituting this expression the right side of eq. (2.4) and then integrating out $\bar{T}$ in the resulting expression by using a Gamma integral gives,

$$|\bar{P}|^{\beta'-\rho_2} \frac{\tilde{\Gamma}_p(\beta'-\rho_1)\tilde{\Gamma}_p(\gamma')\tilde{\Gamma}_p(\gamma)\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\alpha-\rho_1-\rho_2)\tilde{\Gamma}_p(\beta-\rho_1)}{\tilde{\Gamma}_p(\beta')\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)\tilde{\Gamma}_p(\gamma-\rho_1)\tilde{\Gamma}_p(\gamma'-\rho_2)} \quad \ldots (2.7)$$

Now, Taking the M-transform of the left side of eq. (2.4) with respect to the variables $\bar{X}$ and $\bar{Y}$ and the parameters ρ_1 and ρ_1 respectively, we get,

$$\int_{\bar{X}>0}\int_{\bar{Y}>0} |\bar{X}|^{\rho_1-p} |\bar{Y}|^{\rho_2-p} |\bar{P}|^{-\beta'} \times \bar{F}_2(\alpha,\beta,\beta';\gamma,\gamma';-\bar{X}-\bar{P}^{-1/2}\bar{Y}\bar{P}^{-1/2})d\bar{X}\,d\bar{Y} \quad \ldots (2.8)$$

and then using the definition (1.2) yields the same result as in eq. (2.7) above.

THEOREM 2.3:

$$|\bar{F}_3|(\alpha,\alpha',\beta,\beta';\gamma,\gamma';-\bar{X},-\bar{Y})$$

$$= \frac{\tilde{\Gamma}_p(\gamma+\gamma')}{\tilde{\Gamma}_p(\gamma)\tilde{\Gamma}_p(\gamma')} \int_0^I |\bar{U}|^{\gamma-P} |I-\bar{U}|^{\gamma'-P} {}_2\bar{F}_1(\alpha,\beta;\gamma;-\bar{U}^{1/2}\bar{X}\bar{U}^{1/2}) \times$$

$${}_2\bar{F}_1(\alpha',\beta';\gamma';-(I-\bar{U}^{1/2}\bar{Y}(I-\bar{U}^{1/2})d\bar{U} \quad \ldots (2.9)$$

for Re $0 < \bar{U} < I$ and for re $(\gamma, \gamma') > p-1$.

PROOF: Taking the M-transform of the right side of eq. (2.9) with respect to the variables $\bar{X}$ and $\bar{Y}$ and the parameters ρ_1 and ρ_1 respectively, we get,

$$\int_{\overline{X}>0}\int_{\overline{Y}>0}|\overline{X}|^{\rho_1-p}|\overline{Y}|^{\rho_2-p}\ {}_2\overline{F}_1(\alpha,\beta;\gamma;-\overline{U}^{1/2}\overline{X}\,\overline{U}^{1/2})\times$$

$$ {}_2\overline{F}_1(\alpha',\beta';\gamma';-(I-\overline{U}^{1/2}\overline{Y}(I-\overline{U}^{1/2})\,d\overline{X}\,d\overline{Y} \qquad \ldots(2.10)$$

Applying the transformations,

$\overline{X}_1\ \overline{U}^{1/2} = \overline{X}\ \overline{U}^{1/2})\ \overline{Y}_1 = (I-\overline{U})^{1/2}\overline{Y}(I-\overline{U}^{1/2}$ with $d\overline{X}_1 = |\overline{U}|^p\, d\overline{X}$,

$d\overline{Y}_1 = (I-\overline{U})^p\, d\overline{Y}$ and $|\overline{X}_1| = |\overline{U}|\,|\overline{X}|$, $|\overline{Y}_1| = |I-\overline{U}|\,|\overline{Y}|$

to the above expression and then applying eq. (2.3.5) page 38 of Mathai [4] leads us to,

$$|\overline{U}|^{-\rho_1}\ |I-\overline{U}|^{-\rho_1}\ \frac{\tilde{\Gamma}_p(\gamma)\tilde{\Gamma}_p(\alpha'-\rho_1)\tilde{\Gamma}_p(\beta'-\rho_2)\tilde{\Gamma}_p(\gamma)}{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)\tilde{\Gamma}_p(\gamma-\rho_1)\tilde{\Gamma}_p(\alpha)}\times$$

$$\frac{\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\alpha-\rho_1)\tilde{\Gamma}_p(\beta-\rho_1)}{\tilde{\Gamma}_p(\beta')\tilde{\Gamma}_p(\gamma'-\rho_2)} \qquad \ldots(2.11)$$

Substituting this expression on the right side of eq. (2.9) and then integration out the variable $\overline{U}$ in the resulting expression by using a type – 1 Beta integral results in M($\overline{F}_3$) (definition (1.3)).

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