APPLICATION TO CONVEX PROGRAMMING

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Introduction- In this chapter we concentrate on properties of convex sets in a Hilbert space and some of the related problems of important in application to convex programming.

Through this chapter we shall assume that all Hilbert spaces are real Hilbert spaces to avoid indicating real part of the inner product every time. As an application of the separation theorem for convex sets, let us now consider a class of convex programming problems, where we seek to minimize convex functional subject to convex inequalities.

Separation Theorem (1.1):

Statement: Let A and B be convex subsets of a finite dimensional real Hilbert space. If one of the two conditions $A \bigcap B$ is void.A has a non-empty interior and B does not intersect the interior of A hold. Then A and B can be separated by a hyper plane: that is to say there is a non-zero vector v such that $\sup_{x \in A}[v, x] \leq c \leq \underset{y \in B}{Inf}[v, y]$ and the separating hyper plane then may be taken as all x satisfying $[v, x] = C$

Proof: Let us denote by $(A - B)$ the set consisting of all points of the form Then $(A - B)$ is clearly convex. Moreover zero is not an interior point of $(A - B)$ Thus requires proof only under (ii). Suppose then (ii) holds and the origin is an interior point of $(A - B)$.Then $U_{t \geq 0} t(A - B)$ must be the whole space clearly.

Let x_0 be an interior point of A and y any point of B.

Then $y - x_0 = t(u - v), t > 0; u \in A; v \in B$

$$\text{Or, } \frac{y + tv}{(1+t)} = \frac{x_0 + tu}{(1+t)}$$

But every point on the line segment joining x_0 and u except possible for u is an interior point of A. And the line segment joining y and v is in B by convexity. Hence B intersects the interior of A which is a contradiction.

Suppose first that zero is an interior point of the complement of $(A-B)$. Then we can take the projection of zero on the closure of $(A-B)$ and denoting it by Z, We have $[-z, z-y] \geq 0$ for every y in $(A-B)$ or, $[z, y] \geq [z, z] > 0$ and

Hence $\underset{x\in A}{Inf}[z, x] > \underset{y\in B}{\sup}[z, y]$

As required, now if zero is not an interior point of the complement of $(A-B)$ it must be boundary point of $(A-B)$ and hence we can find a sequence of points x_n not in the closure of $(A-B)$ converges to zero. Now denoting the projection of x^n on the closure of $(A-B)$ by Z_n

we have $[x_n - z_n, z_n - y] \geq 0, y \in (A-B)$

And we can clearly take $e_n = -(x_n - z_n)/\|x_n - z_n\|$ which will be a unit vector.

Now since $[z_n - x_n, z_n - x_n] \geq 0$

we can as $[z_n - x_n, y] \geq [z_n - x_n, z_n]$

$$\geq [z_n - x_n, x_n]$$

$or [e_n, y] \geq [e_n, z_n] \geq [e_n, x_n]$ For every y in $(A-B)$

Hence we have demonstrated a support plane through each of sequence of points which are boundary point namely the Z_n and which converges to the given boundary point, the latter since $\|0 - z_n\| \leq \|0 - x_n\| + \|x_n - z_n\|$

$$\leq \|0 - x_n\| + \|0 - x_n\| \to 0$$

Now of the arguments so far has used the restriction to finite dimensions and hence will hold in infinite dimensions. The next step will need the finite dimensional set-up by the Bolzano-Weiesstrass theorem the bounded set of points e_n must have a limit point. Denote if be e_0 then e_0 must actually be a unit vector. And taking the limit in we obtain

$[e_0, y] \geq [e_0, 0], y \in (A-B)$

Or, there is a support plane through the boundary point zero we obtain

$$\underset{x\in A}{Inf}\left[e_0, x\right] \geq C \geq \sup_{y\in B}\left[e_0, y\right]$$

Example (1.2): The following example shows that this result cannot be improved. Thus we shall demonstrate a convex set without support planes through some of its boundary points. Thus let $H =$ the space of square sum able real sequences. Consider the positive cone, that is the class C of square sum able sequences with all terms non-negative. C is clearly convex and closed. It is readily verified that C contains no interior point that every point of C is a boundary point. We claim now that any point with the property that every term in the corresponding sequence is actually bigger than zero cannot have a support plane through it for let Z be such a point. Support for some h in H.

$\left[h, x\right] \leq \left[h, z\right]$ For every x in C

Since for any positive number λ, we must have $\lambda\left[h, z\right] \leq \left[h, z\right]$ it follows that

$\left[h, z\right] \leq 0$ Taking $\lambda \to 0$

$\left[h, z\right] \geq 0$ Taking $\lambda \to 0$

Or, $\left[h, z\right] = 0$

Clearly the sequence corresponding to h cannot contain positive terms then h must actually be zero, since no term of Z is zero. Note that there is indeed a support plane through every point in C corresponding to sequences in which at least one term is zero and clearly such points are dense in C.

Convex Programming

As an application of the separation theorem for convex sets, let us now consider a class of convex programming problems, where we seek to minimize convex functional subject to convex inequalities. Let us finish note a property of continuous convex functional over a Hilbert space.

Theorem (1.3): A continuous convex functional defined on a Hilbert space achieves its minimum on every convex closed bounded set.

Proof: If the space is finite-dimensional we do not need the condition that the set is convex. In infinite dimensions, we note that if $\{x_n\}$ is a minimizing sequences, then since the sequence is bounded, we may work with a weakly convergent subsequence and we have weak lower semi discontinuity $\lim f\left(x_n\right) \geq f\left(x\right)$ where x is the weak limit so that the minimum is the minimum is equal to $f\left(x\right)$. Since a strongly closed convex set is weakly closed, x belongs to the closed convex set.

We shall now state a basic result characterizing the minimal point of a convex functional subject to convex inequalities; we shall not need to state any continuity properties.

Theorem (1.4): Let $f\left(x\right), f_i\left(x\right)$ be convex functional defined on a convex subset C of a Hilbert space and let it be required to minimize $f\left(.\right)$ on C subject to $f_i\left(x\right) \leq 0,\ i = 1, 2......n$

Let x_0 be a point where the minimum is attained. Assume further that for each nonzero, nonnegative vector u in E_n, there is a point x in C such that $\sum_1^n u_k f_k\left(x\right) < 0, \{u_n\}$ denoting the components of u. Then there exists nonnegative vector v with components $\{v_k\}$ such that $\underset{x\in C}{Min}\left\{ f\left(x\right) + \sum_1^n v_k f_k\left(x\right)\right\} = f\left(x_0\right) + \sum_1^n v_k f_k\left(x_0\right) = f\left(x_0\right)$ moreover, for any nonnegative vector u in E_n. $f\left(x\right) + \sum_1^n v_k f_k\left(x\right) \geq f\left(x_0\right) + \sum_1^n v_k f_k\left(x_0\right)$

$$\geq f\left(x_0\right) + \sum_1^n u_k f_k\left(x_0\right)$$

In other words $\left(x_0, v\right)$ is a saddle point for the function $\phi\left(x,u\right)$ with

$$\phi\left(x,u\right) = f\left(x\right) + \sum_1^n u_k f_k\left(x\right)$$

Where u takes values in the positive cone of E_n and x in C.

Proof: Define the following sets A and B in E_{n+1} ;

$$A = \left\{ y = \left(y_0, y_1.....y_n\right) \in E_{n+1}\ such\quad that\ y_0 \geq f\left(x\right), y_k \geq f_k\left(x\right)\ for\ some\quad x\quad in\quad C, k = 1, 2, ...n\right\}$$
$$B = \left\{ y = \left(y_0, y_1......y_n\right) \in E_{n+1}\ such\quad that\ y_0 < f\left(x_0\right), y_i < 0, i = 1, 2.....n\right\}$$

Then it is readily verified that A and B are convex sets in E_{n+1} and that they are nonintersecting. Hence by theorem (5.14) they can be separated. Hence we can find $v_k, k = 0, 1, 2....n$ such that

$$\underset{x\in C}{Inf}\ v_0 f(x)+\sum_1^n v_k f_k(x) \ge v_0 f(x_0)-\sum_1^n v_k |y_k|$$

Since this must hold for arbitrary $|y_k|$, we must have that $v_k, k=1,2.....n$ are nonnegative. In particular, we have letting $|y_k|$ go to zero

$v_0 f(x_0)+\sum_1^n v_k f_k(x_0) \ge v_0 f(x_0)$ And since $f_k(x_0)$ are negative,

It follows that $\sum_1^n v_k f_k(x_0)=0$

We shall next show that v_0 must be positive (greater than zero).

Now if all $v_k, k=1,2....,n$ are zero, v_0 cannot be zero and from $v_0 z_0 \ge v_0 y_0$ for any $y_0 < f(x_0) \le z_0$, it follows that v_0 must be actually positive.

Suppose then that not all v_k are zero, $k=1,2...,n.$ Then by hypothesis these exists x in C such that $\sum_1^n v_k f_k(x) < 0$.

But for any z_0 greater than or equal to $f(x)$

we must have $v_0(z_0 - f(x_0)) \ge -\sum_1^n v_k f_k(x) > 0$ and hence v_0 must be positive.

Hence dividing by v_0 and still using v_k for $v_k/v_0, k=1,2......,n$

we have

$$f(x)+\sum_1^n v_k f_k(x) \ge f(x_o) = f(x_0)+\sum_1^n v_k f_k(x_0)$$

and easily deduced from this.

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