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SOME VALUABLE RESULTS ON GOLDEN RATIO

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ABSTRACT

Current research paper we discussed some valuable results on golden ratio such as If $\varphi^n = a_{n-1}.\varphi + b_{n-1}$; where $a_{n-1} = K$ and $b_{n-1} = L : \infty > K \geq L \geq 0$ then $\varphi^{n+1} = a_n.\varphi + b_n$; with $a_n = a_{n-1} + b_{n-1}$ and $b_n = a_{n-1}$ $i.e.$ $\varphi^{n+1} = (K+L).\varphi + K$ with $a_n = a_{n-1} + b_{n-1} = K + L$ and $b_n = a_{n-1} = K$ $\forall\, n \in \mathrm{N}$.

Key Words: Golden Ratio, Fibonacci sequence, Conjugate, and Golden Rectangle.

Introduction: Mathematician Mark Barr proposed using the first letter in the name of Greek sculptor Phidias, phi, to symbolize the golden ratio. Usually, the lowercase form (φ) is used. Sometimes, the uppercase form (ϕ) is used for the reciprocal of the golden ratio, $1/\varphi$.[1]

The golden ratio has fascinated Western intellectuals of diverse interests for at least 2,400 years. Some of the greatest mathematicians of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. But the fascination with the Golden Ratio is not confined just to mathematicians. Biologists, artists, musicians, historians, architects, psychologists, and even mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.[2]

Ancient Greek mathematicians first studied what we now call the golden ratio because of its frequent appearance in geometry. The division of a line into "extreme and mean ratio" is important in the geometry of regular pentagrams and pentagons. Euclid's Elements provides the first known written definition of what is now called the golden ratio: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser." Euclid explains a construction for cutting a line "in extreme and mean ratio", i.e., the golden ratio.[3] Throughout the Elements, several propositions and their proofs employ the golden ratio.[4]

The first known approximation of the golden ratio and its inverse by a decimal fraction, given by Michael Maestlin of the University of Tübingen in 1597 in a letter to his former student Johannes Kepler.[5]

Mathematicians since Euclid have studied the properties of the golden ratio, including its appearance in the dimensions of a regular pentagon and in a golden rectangle, which may be cut into a square and a smaller rectangle with the same aspect ratio. The golden ratio has also been used to analyze the proportions of natural objects as well as man-made systems such as financial markets, in some cases based on dubious fits to data.[6]

Continuing the above discussion in this research paper I try to establish some valuable results on the golden ratio. These results establish the new dimensions in the study of the golden ratio. The golden ratio herein is defined as-

Definition (1): Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. Expressed algebraically, for quantities a and b with $a > b > 0$,

$$\frac{a+b}{a} = \frac{a}{b}$$

and it is denoted by the Greek letter phi(φ).

Result (1): From the definition of golden ratio

$$\frac{a+b}{a} = \frac{a}{b} = \varphi \Rightarrow 1 + \frac{b}{a} = \frac{a}{b} \Rightarrow 1 + \frac{1}{\varphi} = \varphi \Rightarrow 1 + \varphi = \varphi^2$$

Theorem (1): The golden ratio φ is the root of the quadratic equation $x^2 - x - 1 = 0$.

Proof: Since we know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given as

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ [By quadratic formula]

Therefore the roots of the quadratic equation $x^2 - x - 1 = 0$ are

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\Rightarrow$ the two roots of the quadratic equation $x^2 - x - 1 = 0$ are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

$\Rightarrow$one root of the quadratic equation $x^2 - x - 1 = 0$ is $\frac{1+\sqrt{5}}{2}$ which is known as the golden ratio.

[From the Definition (1)]

Result (2): The value of golden ratio $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887 \dots \dots \dots \dots \dots$

Definition (2): The reciprocal root of the quadratic equation for golden ratio φ is

$$\frac{1}{\varphi} = \varphi^{-1} = \frac{2}{1+\sqrt{5}} = \left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]\left[\frac{2}{1-\sqrt{5}}\right] = \frac{1-\sqrt{5}}{2}$$

The absolute value of this quantity (≈ **0.618**) corresponds to the length ratio taken in reverse order and is sometimes referred to as the golden ratio conjugate.[1] It is denoted here by the capital Phi (Φ):

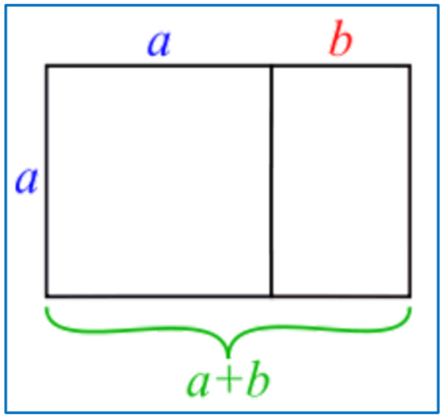
$$\Phi = \frac{1}{\varphi} = \varphi^{-1} = 0.6180339887 \dots \dots \dots \dots \dots$$

Result (3): The golden ratio among positive numbers has unique property, that

$\frac{1}{\varphi} = \varphi - 1$ and it's inverse:

$$\frac{1}{\Phi} = \Phi + 1$$

This means **0.61803...:1 = 1:1.61803....**



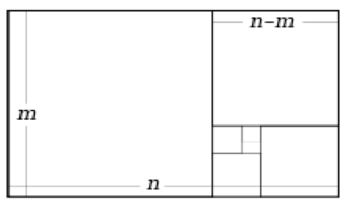
Definition (3):

Figure 3.1

A golden rectangle with longer side a and shorter side b, when placed adjacent to a square with sides of length a [figure 3.1], will produce a similar golden rectangle with longer side $a + b$ and shorter side a. This illustrates the relationship $\frac{a+b}{a} = \frac{a}{b} = \varphi$.

Theorem (2): The golden ratio is an irrational number.

Proof:



If φ were rational, then it would be the ratio of sides of a rectangle with integer sides (the rectangle comprising the entire diagram). But it would also be a ratio of integer sides of the smaller rectangle (the rightmost portion of the diagram) obtained by deleting a square. The sequence of decreasing integer side lengths formed by deleting squares cannot be continued indefinitely because the integers have a lower bound, so φ cannot be rational.

Recall that: the whole is the longer part plus the shorter part; the whole is to the longer part as the longer part is to the shorter part.

If we call the whole n and the longer part m, then the second statement above becomes n is to m as m is to $n - m$, or, algebraically

$$\frac{n}{m} = \frac{m}{n-m} \qquad [2.1]$$

To say that φ is rational means that φ is a fraction n/m where n and m are integers. We may take n/m to be in lowest terms and n and m to be positive. But if n/m is in lowest terms, then the identity [2.1] says $m/(n - m)$ is in still lower terms. That is a contradiction that follows from the assumption that φ is rational.

Alternate Proof: If $\frac{1+\sqrt{5}}{2}$ is rational, then $2\left[\frac{1+\sqrt{5}}{2}\right] - 1 = \sqrt{5}$ is also rational, which is a contradiction if it is already known that the square root of a non-square natural number is irrational.

Result (4): The formula $\varphi = 1 + 1/\varphi$ can be expanded recursively to obtain a continued fraction for the golden ratio:

$$\varphi = [1; 1, 1, , 1 \ldots \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

and it's reciprocal:

$$\varphi^{-1} = [1; 1, 1, , 1 \ldots \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

The convergent of these continued fractions (1/1, 2/1, 3/2, 5/3, 8/5, 13/8... or 1/1, 1/2, 2/3, 3/5, 5/8, 8/13 ...) are ratios of successive Fibonacci numbers.

Result (5): The equation $\varphi^2 = 1 + \varphi$ likewise produce the continued square root, or infinite surd, form: $\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}}}$

Result (6): The Fibonacci sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987...

The closed-form expression for the Fibonacci sequence involves the golden ratio:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^n}{\sqrt{5}}$$

The golden ratio is the limit of the ratios of successive terms of the Fibonacci sequence as originally shown by Kepler:

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \varphi$$

Therefore, if a Fibonacci number is divided by its immediate predecessor in the sequence, the quotient approximates φ ; e.g., 987/610 ≈ 1.6180327868852. These approximations are alternately lower and higher than φ , and converge on φ as the Fibonacci numbers increases, and:

$$\sum_{n=1}^{\infty} | F(n)\,\varphi - F(n+1)| = \varphi$$

More generally:

$$\lim_{n\to\infty} \frac{F\,(n+k)}{F\,(n)} = \varphi^k$$

where above, the ratios of consecutive terms of the Fibonacci sequence, is a case when $a = 1$.

Furthermore, the successive powers of φ obey the Fibonacci recurrence:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

This identity allows any polynomial in φ to be reduced to linear expressions.

Example (1): $3\varphi^3 - 5\varphi^2 + 4$

$$= 3(\varphi^2 + \varphi) - 5\varphi^2 + 4$$

$$= 3[(\varphi + 1) + \varphi] - 5(\varphi + 1) + 4 = \varphi + 2 \approx 3.618.$$

Theorem (3): Let $\varphi = a_0\varphi + b_0$, where $a_0 = 1$ and $b_0 = 0$ then $\varphi^2 = a_1.\varphi + b_1$; with $a_1 = a_0 + b_0$ and $b_1 = a_0$ $i.e.$ $\varphi^2 = 1.\varphi + 1$ where $a_1 = 1$ and $b_1 = 1$.

Proof: Since $\varphi = a_0\varphi + b_0$, where $a_0 = 1$ and $b_0 = 0$

$\Rightarrow \varphi^2 = \varphi.\varphi = \varphi.(a_0\varphi + b_0) \Rightarrow a_0\varphi^2 + b_0$ -- [3.1]

But we know that the golden ratio φ is the root of the quadratic equation

$\varphi^2 - \varphi - 1 = 0$ -------------------- --- [3.2]

Substitute the value of $\varphi^2 = \varphi + 1$ from equation [3.2] in equation [3.1] we find out that $\varphi^2 = a_0(\varphi + 1) + b_0 \Rightarrow (a_0 + b_0)\,\varphi + a_0 \Rightarrow \varphi^2 = a_1.\varphi + b_1$, where $a_1 = a_0 + b_0$ and $b_1 = a_0 \Rightarrow i.e.$ $\varphi^2 = 1.\varphi + 1$ where $a_1 = 1 + 0 = 1$ and $b_1 = a_0 = 1$.

Result (7): If $\varphi^2 = a_1.\varphi + b_1$; where $a_1 = 1$ and $b_1 = 1$ then $\varphi^3 = a_2.\varphi + b_2$; with $a_2 = a_1 + b_1$ and $b_2 = a_1$ $i.e.$ $\varphi^3 = 2.\varphi + 1$ with $a_2 = 2$ and $b_2 = 1$.

Proof: Since $\varphi^2 = a_1.\varphi + b_1$; where $a_1 = 1$ and $b_1 = 1$

$\Rightarrow \varphi^3 = \varphi.\varphi^2 = \varphi.(a_1\varphi + b_1) \Rightarrow a_1\varphi^2 + b_1$ Substitute the value of $\varphi^2 = \varphi + 1$ from equation [3.2] {theorem (3)} we find out that $\varphi^3 = a_1(\varphi + 1) + b_1 \Rightarrow (a_1 + b_1)\,\varphi + a_1 \Rightarrow \varphi^3 = a_2.\varphi + b_2$; with $a_2 = a_1 + b_1$ and $b_2 = a_1$.

$\Rightarrow \varphi^3 = a_2.\varphi + b_2$; with $a_2 = 1 + 1 = 2$ and $b_2 = 1$ ie $\varphi^3 = 2.\varphi + 1$.

Result (8): If $\varphi^3 = a_2.\varphi + b_2$; where $a_2 = 2$ and $b_2 = 1$ then $\varphi^4 = a_3.\varphi + b_3$; where $a_3 = a_2 + b_2$ and $b_3 = a_2$ ie $\varphi^4 = 3.\varphi + 2$, where $a_3 = 3$ and $b_3 = 2$.

Proof: Since $\varphi^3 = a_2.\varphi + b_2$; where $a_2 = 2$ and $b_2 = 1$

$\Rightarrow \varphi^4 = \varphi.\ \ \varphi^3 = \varphi.(a_2\varphi + b_2) \Rightarrow a_2\varphi^2 + b_2$ Substitute the value of $\varphi^2 = \varphi + 1$ from equation [3.2] {theorem (3)} we find out that $\varphi^4 = a_2(\varphi + 1) + b_2 \Rightarrow (a_2 + b_2)\,\varphi + a_2 \Rightarrow \varphi^4 = a_3.\varphi + b_3$; where $a_3 = a_2 + b_2$ and $b_3 = a_2$.

$\Rightarrow \varphi^4 = a_3.\varphi + b_3$; where $a_3 = 2 + 1 = 3$ and $b_2 = 2$ $i.e.$ $\varphi^4 = 3.\varphi + 2$.

Result (9):{GernelisedResult} If $\varphi^n = a_{n-1}.\varphi + b_{n-1}$; where $a_{n-1} = K$ and $b_{n-1} = L : \infty > K \geq L \geq 0$ then $\varphi^{n+1} = a_n.\varphi + b_n$; with $a_n = a_{n-1} + b_{n-1}$ and $b_n = a_{n-1}$ $i.e.$ $\varphi^{n+1} = (K + L).\varphi + K$ with $a_n = a_{n-1} + b_{n-1} = K + L$ and $b_n = a_{n-1} = K$ $\forall n \in \mathrm{N}$.

Proof: Since $\varphi^n = a_{n-1}.\varphi + b_{n-1}$; where $a_{n-1} = K$ and $b_{n-1} = L : K \geq L \geq 0$

$\Rightarrow \varphi^{n+1} = \varphi.\ \ \varphi^n = \varphi.(a_{n-1}\varphi + b_{n-1}) \Rightarrow a_{n-1}\varphi^2 + b_{n-1}$ Substitute the value of $\varphi^2 = \varphi + 1$ from equation [3.2] {theorem (3)} we find out that $\varphi^{n+1} = a_{n-1}(\varphi + 1) + b_{n-1} \Rightarrow (a_{n-1} + b_{n-1})\,\varphi + a_{n-1} \Rightarrow \varphi^{n+1} = a_n.\varphi + b_n$; where $a_n = a_{n-1} + b_{n-1}$ and $b_n = a_{n-1}$.

$\Rightarrow \varphi^{n+1} = a_n.\varphi + b_n$; where $a_n = K + L$ and $b_2 = K$ $i.e.$ $\varphi^{n+1} = (K + L).\varphi + K$.

Conclusion: We conclude that "If $\varphi^n = a_{n-1}.\varphi + b_{n-1}$; where $a_{n-1} = K$ and $b_{n-1} = L : \infty > K \geq L \geq 0$ then $\varphi^{n+1} = a_n.\varphi + b_n$; with $a_n = a_{n-1} + b_{n-1}$ and $b_n = a_{n-1}$ $i.e.$ $\varphi^{n+1} = (K + L).\varphi + K$ with $a_n = a_{n-1} + b_{n-1} = K + L$ and $b_n = a_{n-1} = K$ $\forall n \in \mathrm{N}$ ".

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