International Research Journal of Mathematics, Engineering and IT
Vol. 2, Issue 8, August 2015 IF- 2.868 ISSN: (2349-0322)

Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

A PROMINENT OBSERVATION ON A NUMBER $N = p^q q + 1$

Rajiv Kumar
Assistant Professor
Department of Mathematics D. J. College, Baraut (Baghpat)

ABSTRACT

In this paper we discuss some prominent observation on a number $N = p^q q + 1$, *where* p *and* q *are prime numbers. Here I find out that* N *is composite in nature for all* $p \& q$: $p \neq q \neq 2$, $p = q \neq 2$ & $p = q = 2$ *and N is mixed in nature for* $p = 2 \neq q$ *.*

Key Words: Composite Number, Co- prime Number, Divisor, Greatest Common Divisor, and Prime Number.

Introduction:

There are a multitude of conjectures in the theory of prime numbers. For example, it is not known if there are infinitely many primes of the form $N^2 + 1$. On the other hand we find out that the linear polynomial $an + b$ always represents an infinite number of primes when $(a, b) = 1$. This is the celebrated theorem of Dirichlet on primes in an arithmetic progression [4].

It is not known whether there exist infinitely many primes of the form $2^N + 1$, the so called Fermat primes, or if there are infinitely primes of the form $2^N - 1$, the Mersenne primes [1]. That is we know that the number of the form $N^2 + 1$, $2^N + 1$, and $2^N - 1$ are mixed in nature. Continuing the above discussion we discuss some remarkable observation on a number of the form $N = p^q q + 1$. Going on Main part of the paper here I introduce some famous definitions, results and theorems.

Definition (1): A **prime number** (or a **prime**) is a natural number greater than 1 that has no positive divisors other than 1 and itself.

Example (1): The smallest prime is 2 and all the primes 3, 5, 7, 11... etc. are odd.

Definition (2): A natural number greater than 1 that is not a prime number is called a **composite number.**

Result (1): Each composite number has the form $n = a\,b$ where $1 < a < n$ and $1 < b < n$.

Example (2): The number 6, 8, 12...... etc. are composite. From the result (1) we can saw that each given integer can be written in the form $n = a\,b$ where $1 < a < n$ and $1 < b < n$.

Definition (3): For integers m *and* n with $m \neq 0$ it is said that m *divides* n if n is a **multiple** of m, that is, if there is an integer k so that $n = mk$. If m *divides* n, we can write

$$m|n$$

Result (2): Every integer divides **0** (since $0 = 0\,b$ *for all* b), **1** divides every integer and every integer divides itself.

Definition (4): Let a *and* b be any given integers not all zero. Then a positive integer d possessing the following two properties is called the greatest common divisor of them.

(i) $d|a$ *and* $d|b$.

(ii) If any integer c divides a, and b then c divides d.

The g. c. d. of a and b is very often written as (a, b).

Result (2): Two integers a *and* b are co-prime (or relatively prime) if g. c. d. $(a, b) = 1$.

Theorem (1): (Euclid's Theorem) There exist infinitely many primes.

Proof: Suppose there exist only finite number of primes namely $p_{1,}$.................,p_k in ascending order. Then consider the integer $N =$ $p_{1,}$.................,$p_k + 1$. It is obvious that $N > p_k$. If $N$is a prime, then it is clear that there exists a prime greater than p_k. On the other hand if N is composite, it is not divisible by any of the primes $p_{1,}$.................,p_k because such a division leaves 1 as remainder. Hence N , being composite must be divisible by a prime greater than p_k . Thus in either case there exist a prime greater than p_k . But this is contradicts our assumption that there are only a finite number of primes. It follows that there are infinitely many primes. [5]

Theorem (2): There are many infinitely many primes of the form $4k + 3$.

Proof: This proof is by contradiction. Suppose that there are only finitely many primes of this form, say $p_{1,\ldots\ldots\ldots,}p_k$. Let $m = 4p_{1,\ldots\ldots\ldots,}p_k - 1$, so m has the form $4k + 3$ (with $q = p1,\ldots\ldots\ldots,pk-1$). Since m is odd, so it is divisible by each prime p . So p has the form $4k + 1$ or $4k + 3$ for some k . If each such p has the form $4k + 1$, then m must also have this form, which is false. Hence m must be divisible by at least one prime p of the form $4k + 3$. By our assumption, $p = p_i$ for some i , so p divides $4p_{1,\ldots\ldots\ldots,}p_k - m = 1$, which is impossible. This contradiction proves the result. [2]

Theorem (3): There are many infinitely many primes of the form $4k + 1$.

Proof: Assume that there are a finite number of primes of the form$4k + 1$. Let these be 5, 13, 17, 19... q where q is the largest such prime. Then consider the integer

$$N = (2 \times 5 \times 13 \times \ldots\ldots\ldots \times q)^2 + 1$$

$$= (m)^2 + 1 \quad \text{Say.}$$

If N is a prime, then it is a prime of the form $4k + 3$ because $N > q$. On the other hand if N is composite, it is not divisible by any of the primes$2, 5, 13, 17, \ldots\ldots\ldots\ldots, q$. Therefore it is divisible by a prime of the form $4k + 3$. Thus in both the cases there exists a number m such that $(m)^2 + 1$ is divisible by a prime of the form $4k + 3$. But this is contradicts corollary 6.3.2. Hence our assumption is wrong and the theorem is true. [5]

Theorem (4): If $p = q = 2$ than N is composite in nature.

Proof: we know that $N = p^q\, q + 1$. If $p = q = 2 \Rightarrow N = 2^2 2 + 1 = 8 + 1 = 9$ which is a composite number. Hence we can say that if $p = q = 2$ than N is composite in nature.

Theorem (5): If $p = q \neq 2$ than N is composite in nature.

Proof: we know that $N = p^q q + 1$. If $p = q \Rightarrow N = p^p p + 1 = p^{p+1} + 1$. Also given that $p \neq 2$, that is , p is an odd prime number $\Rightarrow p^{p+1}$ is also an odd number. Therefore $N = p^{p+1} + 1$ is an even number › 2. Since we know that every even number › 2 is a composite number, hence we can say that if $p = q \neq 2$ than N is composite in nature.

Theorem (6): If $p \neq q \neq 2$ than N is composite in nature.

Proof: we know that $N = p^q q + 1$. If $p \neq q \neq 2$ ⇒both $p\ and q\ are\ odd\ primes\ with (p, q) = 1$. Since both $p\ and q\ are\ odd\ primes\ with (p, q) = 1$ therefore p^q an odd number⇒ $p^q q$ is also an odd number (the multiplication of two odd numbers is again an odd number).

Therefore $N = p^q q + 1$ is an even number › 2. Since we know that every even number › 2 is a composite number, hence we can say that if $p = q \neq 2$ than N is composite in nature.

Theorem (7): If $p = 2 \neq q$ than N is mixed in nature.

Proof: we know that $N = p^q q + 1$. If $p = 2 \neq q$ ⇒$q\ an\ odd\ prime$.

Since $p = 2$ therefore $N = 2^q q + 1$. Since q is an odd prime therefore it is either of the $4k + 1$ form or $4k + 3$ form. Now we consider the number $2^q q$ which is an even number (the multiplication of an even number and an odd number is again an even number). Since we know that every even number is either of the form $4k$ or form $4k + 2$. Here the number $2^q q$ is only of the form $4k$ then $N = 2^q q + 1$ is of the form $4k + 1$. Since we know that there are many infinitely prime numbers of the form $4k + 1$ as well as many infinitely composite numbers of the form $4k + 1$. Hence we can say that If $p = 2 \neq q$ than N is mixed in nature.

Conclusion:

From the theorem (4) to (7) we conclude that a number $N = p^q q + 1$, $where\ p\ and\ q$are prime numbers is composite in nature for all $p\ \&\ q : p \neq q \neq 2, p = q \neq 2\ \&\ p = q = 2$ and mixed in nature for $p = 2 \neq q$.

References:

1. David M. Burton, Elementary Number Theory, Tata McGraw Hill Private Limited, New Delhi, 2008.
2. Gareth A. Jones and J. Mary Jones, Elementary Number Theory, Springer-Verlag London, 2006.
3. Ivan Niven, Herbert S. Zuckerman and Hugh L. Montgomery, An Introduction To The Theory Of Numbers , John Wiley & Sons, Inc., 2004.

4. Kenneth Ireland Michael Rosen: A Classical Introduction to Modern Number Theory, Springer-Verlag New York, 2005.
5. S. G. Telang; Tata McGraw- Hill Publishing Company Limited; New Delhi, 2001.