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# ON SUMMABILITY FACTORS FOR GENERALIZED ABSOLUTE NÖRLUND SUMMABILITY

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## ABSTRACT

In this paper we determine the necessary and sufficient conditions to obtain summability factors in transition from $|N, p_n, q_n|$ boundedness to $\varphi - |\bar{N}, q_n, \delta|_k$ summability. The result obtained here generalizes many known results.

**AMS subject classification**: 40F05, 40D25

**Key words**: Absolutely Summability Factors, Regular summability methods, Infinite series, Generalized Nőrlund means..

## 1. Introduction

Let $\{U_n\}$ denote the sequence of $|\bar{N}, q_n|$ mean of a sequence $\{s_k\}$. Then

$$U_n = (Q_n)^{-1} \sum_{m=0}^{n} q_m s_m \qquad (Q_n \neq 0) \tag{1.11}$$

where $Q_n = \sum_{m=0}^{n} q_m$.

We say that a sequence $\{s_n\}$ is summable by the method $\varphi - |\bar{N}, p_n, \delta|_k$ if

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |U_n - U_{n-1}|^k < \infty \tag{1.12}$$

We write formally the constants $b_n$ to define by the identity

$$\left(\sum_{m=0}^{n} q_m x^m\right)^{-1} = \sum_{n=0}^{\infty} b_n x^n. \tag{1.13}$$

We also write

$$B_n = b_0 + b_1 + \cdots + b_n. \qquad \textbf{(1.14)}$$

Let $\sum a_n$ be an infinite series with the sequence of partial sum $s_n$. Let $\{u_n\}$ denote the sequences of $(N, p, q)$ means of $\{s_n\}$. Thus

$$u_n = (R_n)^{-1} \sum_{m=0}^{n} p_{n-m} q_m s_m, \qquad \textbf{(1.15)}$$

where

$$R_n = \sum_{m=0}^{n} p_{n-m} q_m, \text{ for any n}, \qquad \textbf{(1.16)}$$

$p_{-1} = q_{-1} = R_{-1} = 0.$

The method (N, p, q) is regular if and only if the following conditions are satisfied:

(i) $\text{Lim}_n\, p_{n-m}q_m / R_n = 0$ for each m,

(ii) $\sum_{m=0}^{n} |p_{n-m} q_m| < M$. where M is a positive constant independent of n.

We say that $\{p_n\} \in \mathcal{M}$, if

$$p_n > 0, \quad \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \quad n = 0, 1, \cdots$$

A series $\sum a_n$ is said to be $(N, p, q)$ bounded, or $\sum a_n = O(1)\ (N, p, q)$ if

$$\sum_{m=1}^{n} p_{n-m} q_m = O(R_n), \text{as } n \to \infty.$$

For two methods of summability $A$ and $B$, we say that $\varepsilon_n \in (A, B)$ if $\sum a_n \varepsilon_n$ is summable by method B whenever $\sum a_n$ is summable by the method A.

## 2. Known results.

The following theorem was proved by Das in 1966 [1].

**Theorem 2.1.** Let $\{p_n\} \in \mathcal{M}, q_n \geq 0$. Then if $\sum a_n$ is $|N, p, q|$ summable, it is $|\bar{N}, q_n|$ summable.

Later, Singh and Sharma [2] proved the following result.

**Theorem 2.2.** Let $\{p_n\} \in \mathcal{M}$, $q_n > 0$ and let $\{q_n\}$ be a monotonic nondecreasing sequence. Then a necessary and sufficient condition that $\sum a_n \varepsilon_n$ is summable $|\bar{N}, q_n|$ whenever

$$\sum a_n = O(1)\ (N, p, q),$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\varepsilon_n| < \infty,$$

$$\sum_{n=0}^{\infty} |\varepsilon_n| < \infty, \text{ and}$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \varepsilon_n| < \infty,$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n||\varepsilon_n| < \infty.$$

Recently Rhoades and Savaş [3] generalized the above result in the following form:

**Theorem 2.3.** Let $\{p_n\} \in \mathcal{M}$, $q_n > 0$, $\{q_n\}$ be a monotonic nonincreasing sequence and $nq_n = O(Q_n)$. A necessary and sufficient condition that $\varepsilon \in (|N,p,q|_k, |\bar{N},q_n|_k)$ whenever

(i) $\sum a_n = O(1) |N, p, q|_k$,

(ii) $\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\varepsilon_n|^k < \infty$,

(iii) $\sum_{n=0}^{\infty} \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta\varepsilon_n|^k < \infty$,

(iv) $\sum_{n=0}^{\infty} \left(\frac{Q_{n+1}\Delta q_n}{q_n q_{n+1}}\right)^k \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta\varepsilon_n|^k < \infty$,

(v) $\sum_{n=0}^{\infty} \left(\frac{Q_{n+1}}{q_{n+1}}\right)^k \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta^2\varepsilon_n|^k < \infty$,

is that

(vi) $\sum_{n=1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n}\right)^k |s_n|^k |\varepsilon_n|^k < \infty$

In this note we generalize the above result for $\varphi - \left|\bar{N}, q_{n,}\delta\right|_k$ summability methods, where $\varphi = \{\varphi_n\}$ is a sequence of positive numbers. The result obtained here will generalize many known results.

## 3. Some Lemmas.

We shall require the following lemmas for the proof of our main result.

**Lemma 3.1.**[4] Let $\{p_n\} \in \mathcal{M}$, then

(a) $b_0 > 0,\ b_n \leq 0, n = 1, 2, \cdots$

(b) $\sum_{n=0}^{\infty} b_n x^n$ is absolutely convergent for $|x| \leq 1$, and

(c) $\sum_{n=0}^{\infty} b_n = 0$, when $\sum_{n=0}^{\infty} p_n = \infty$,

$> 0$, otherwise.

**Lemma 3.2.** [1] If

$$u_n = (R_n)^{-1} \sum_{m=0}^{n} p_{n-m} q_m s_m,$$

then

$$s_n = (q_n)^{-1} \sum_{m=0}^{n} b_{n-m} R_m s_m.$$

**Lemma 3.3. [1]** We have

$$\sum_{m=0}^{n} B_{n-m} R_m = Q_n,$$

where $B_n$, $R_n$ and $Q_n$ are defined as in Section 1.

## 4. Main Result.

In what follows, we shall prove the following result.

**Theorem 4.1**: Let $\{p_n\} \in \mathcal{M}$, $q_n > 0$, $\{q_n\}$ be a monotonic nonincreasing sequence and $\varphi_n q_n = O(Q_n)$, where $\varphi_n$ is a sequence of positive numbers. A necessary and sufficient condition that $\sum a_n \varepsilon_n$ is summable $\varphi - |\bar{N}, q_n, \delta|_k$, whenever conditions (i) – (v) of Theorem 2.3 hold together with

(vi) $\sum_{n=m+1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-1} = O(Q_m^{-1})$

(vii) $\sum_{n=0}^{\infty} \left(\varphi_n^{\delta k/(k-1)} Q_n q_n^{-1}\right)^{k-1} |\Delta \varepsilon_n|^k < \infty$

is that

(viii) $\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (q_n Q_n^{-1})^k |s_n|^k |\varepsilon_n|^k < \infty$

**Proof**. Let $u_n$ be defined by (1.14) and let $U_n$ be the $|\bar{N}, q_n|$ mean of the sequence $\sum a_n \varepsilon_n$. Then by definition, we have

$$U_n = (Q_n)^{-1} \sum_{j=0}^{n} q_j \sum_{l=0}^{j} a_l \varepsilon_l = (Q_n)^{-1} \sum_{j=0}^{n} (Q_n - Q_{n-1}) a_j \varepsilon_j.$$

Then for $n \geq 1$, we have

$$\begin{aligned} U_n - U_{n-1} &= q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n} Q_{j-1} a_j \varepsilon_j, \\ &= q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n} Q_{j-1} \varepsilon_j \left[\sum_{l=0}^{j} a_l - \sum_{l=0}^{j-1} a_l\right] \\ &= q_n (Q_n Q_{n-1})^{-1} \left[\sum_{j=0}^{n} Q_{j-1} \varepsilon_j \sum_{l=0}^{j} a_l - \sum_{j=0}^{n} Q_j \varepsilon_{j+1} \sum_{l=0}^{j} a_l\right] \\ &= q_n (Q_n Q_{n-1})^{-1} \left[\sum_{j=0}^{n-1} \Delta\left(Q_{j-1} \varepsilon_j\right) \sum_{l=0}^{j} a_l + s_n \varepsilon_n Q_{n-1}\right] \\ &= -q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} q_j s_j \varepsilon_j + q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} Q_j s_j \Delta \varepsilon_j + \\ &\quad + q_n Q_n^{-1} s_n \varepsilon_n \\ &= U_{n1} + U_{n2} + U_{n3}, \text{ say} \end{aligned} \qquad \textbf{(4.11)}$$

Now invoking Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{ni}|^k < \infty, \qquad i = 1, 2, 3. \qquad \textbf{(4.12)}$$

Now, using Lemma 3.32 and Lemma 3.33, we obtain

$$\begin{aligned} U_{n1} &= -q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} q_j s_j \varepsilon_j \\ &= -q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} q_j \varepsilon_j q_j^{-1} \sum_{m=0}^{j} b_{j-m} R_m u_m \\ &= -q_n (Q_n Q_{n-1})^{-1} \sum_{m=0}^{n-1} R_m u_m \sum_{j=m}^{n-1} b_{j-m} \varepsilon_j \\ &= -q_n (Q_n Q_{n-1})^{-1} \sum_{m=0}^{n-1} R_m u_m \left[\sum_{j=m}^{n-1} (B_{j-m} \Delta \varepsilon_j) + b_{n-1-m} \varepsilon_j\right] \\ &= U_{n1}^1 + U_{n1}^2, \text{ say.} \end{aligned} \qquad \textbf{(4.13)}$$

Now, we have

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{n1}^1|^k = \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} [q_n (Q_n Q_{n-1})^{-1}]^k \left|\sum_{m=0}^{n-1} R_m u_m \sum_{j=m}^{n-1} B_{j-m} \Delta\varepsilon_j\right|^k$$

$$\leq \sum_{n=1}^{\infty} (\varphi_n q_n Q_n^{-1})^{k-1} \left(\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}\right)\left(\sum_{m=0}^{n-1} R_m u_m \sum_{j=m}^{n-1} B_{j-m} |\Delta\varepsilon_j|\right)^k$$

$$= O(1) \sum_{n=1}^{\infty}\left(\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}\right) \left(\sum_{j=0}^{n-1} |\Delta\varepsilon_j| \sum_{m=0}^{j} B_{j-m} R_m\right)^k \quad \text{[using (i)]}$$

$$= O(1) \sum_{n=1}^{\infty}\left(\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}\right) \left(\sum_{j=0}^{n-1} |\Delta\varepsilon_j| Q_j\right)^k \quad \text{[using Lemma 3.3]}$$

$$= O(1) \sum_{n=1}^{\infty}\left(\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}\right) \left(\sum_{j=0}^{n-1}|\Delta\varepsilon_j|^k Q_j^k q_j^{1-k}\right) \left(Q_{n-1}^{-1} \sum_{j=1}^{n-1} q_j\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{\infty}\left(\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-1}\right) \left(\sum_{j=0}^{n-1}|\Delta\varepsilon_j|^k Q_j^k q_j^{1-k}\right)$$

$$= O(1) \sum_{j=1}^{\infty}|\Delta\varepsilon_j|^k Q_j^{k-1} q_j^{1-k}\, Q_j \sum_{n=j+1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-1}$$

$= O(1)$ (using (iii) and (vi) **(4.14)**

A similar treatment will yield

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{n1}^2|^k = \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1}\, q_n^k (Q_n Q_{n-1})^{-k} \left|\sum_{m=0}^{n-1} \varepsilon_n B_{n-1-m} R_m u_m\right|^k$$

$$\leq \sum_{n=1}^{\infty} (\varphi_n q_n Q_n^{-1})^{k-1} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k})\left(\sum_{m=0}^{n-1} |\varepsilon_n| B_{n-1-m} R_m\right)^k$$

[using (i)]

$$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n |\varepsilon_n|^k Q_n^{-1} Q_{n-1}^{-k} \left(\sum_{m=0}^{n-1} B_{n-1-m} R_m\right)^k$$

$= O(1)$[using Lemma 3.3 and (ii)] **(4.15)**

Now ,

$$U_n^2 = q_n (Q_n Q_{n-1})^{-1} \sum_{m=0}^{n-1} Q_m \Delta\varepsilon_m s_m$$

$$= q_n (Q_n Q_{n-1})^{-1} \sum_{m=0}^{n-1} Q_m \Delta\varepsilon_m\, q_m^{-1} \sum_{j=0}^{m} b_{m-j} R_j t_j \quad \text{[using Lemma 3.2]}$$

$$= q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} Q_m q_m^{-1} \Delta\varepsilon_m b_{m-j}$$

$$= -q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} B_{m-j} q_{m+1} q_m^{-1} \Delta\varepsilon_m +$$

$$+ q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{m-1} R_j t_j \sum_{m=j}^{n-1} B_{m-j} Q_{m+1} \Delta(q_m^{-1}) \Delta\varepsilon_m +$$

$$+ q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{m-1} R_j t_j \sum_{m=j}^{n-1} B_{m-j} Q_{m+1} q_{m+1}^{-1} \Delta^2\varepsilon_m +$$

$$+ q_n (Q_n Q_{n-1})^{-1} \sum_{j=0}^{n-1} R_j\, t_j B_{n-1-j} Q_n q_n^{-1} \Delta\varepsilon_n$$

$= U_{n2}^1 + U_{n2}^2 + U_{n2}^3 + U_{n2}^4$, say. **(4.16)**

Using condition (i), (iii) and Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{n2}^1|^k = \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (q_n (Q_n Q_{n-1})^{-1})^k \left|\sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} B_{m-j} q_{m+1} q_m^{-1} \Delta\varepsilon_m\right|^k$$

$$\leq \sum_{n=1}^{\infty} (\varphi_n q_n Q_{n-1}^{-1})^{k-1} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}) \left( \sum_{j=0}^{n-1} R_j u_j \sum_{m=j}^{n-1} |B_{m-j}| q_{m+1} q_m^{-1} | \Delta\varepsilon_m | \right)^k$$

$= O(1) \sum_{n=1}^{\infty} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}) \left( \sum_{j=0}^{n-1} R_j \sum_{m=j}^{n-1} |B_{m-j}| q_{m+1} q_m^{-1} | \Delta\varepsilon_m | \right)^k$

$= O(1) \sum_{n=1}^{\infty} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}) \left( \sum_{m=0}^{n-1} q_{m+1} q_m^{-1} |\Delta\varepsilon_m| \sum_{j=0}^{m} R_j B_{m-j} \right)^k$

$= O(1) \sum_{n=1}^{\infty} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}) \left( \sum_{m=0}^{n-1} q_{m+1} q_m^{-1} |\Delta\varepsilon_m| Q_m \right)^k$

$= O(1) \sum_{n=1}^{\infty} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}) \sum_{m=0}^{n-1} q_{m+1}^k q_m^{-k} |\Delta\varepsilon_m|^k Q_m^k q_m^{1-k} (Q_{n-1}^{-1} \sum_{m=0}^{n-1} q_m)^{k-1}$

$= O(1) \sum_{m=0}^{\infty} q_{m+1}^k q_m^{-k} |\Delta\varepsilon_m|^k Q_m^k q_m^{1-k} \sum_{n=m+1}^{\infty} (\varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1})$

$= O(1) \sum_{m=0}^{\infty} q_{m+1}^k q_m^{-k} |\Delta\varepsilon_m|^k Q_m^k q_m^{1-k} Q_m^{-1}$ [using (vi)]

$= O(1) \sum_{m=0}^{\infty} Q_m^{k-1} q_m^{1-k} |\Delta\varepsilon_m|^k$

$= O(1)$ (using (iii) **(4.17)**

Further, using Hölder's inequality

$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{n2}^2|^k$

$= \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (-q_n (Q_n Q_{n-1})^{-1})^k \left| \sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} B_{m-j} Q_{m+1} \Delta(q_m^{-1}) \Delta\varepsilon_m \right|^k$

$$\leq \sum_{n=1}^{\infty} (\varphi_n q_n Q_n^{-1})^{k-1} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \left( \sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} |B_{m-j}| Q_{m+1} \Delta(q_m) q_m^{-1} q_{m+1}^{-1} |\Delta\varepsilon_m| \right)^k$$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \left( \sum_{m=0}^{n-1} Q_{m+1} \Delta(q_m) q_m^{-1} q_{m+1}^{-1} |\Delta\varepsilon_m| \sum_{j=0}^{m} R_j B_{m-j} \right)^k$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} (\sum_{m=0}^{n-1} Q_m Q_{m+1} \Delta(q_m) q_m^{-1} q_{m+1}^{-1} |\Delta\varepsilon_m|)^k$ [using Lemma 2.3]

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \sum_{m=0}^{n-1} (Q_m Q_{m+1} \Delta(q_m) q_m^{-1} q_{m+1}^{-1} |\Delta\varepsilon_m|)^k$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \sum_{m=0}^{n-1} (Q_m Q_{m+1} q_m^{-1} q_{m+1}^{-1})^k q_m^{1-k} |\Delta q_m|^k |\Delta\varepsilon_m|^k$

$\times (Q_{m-1}^{-1} \sum_{m=0}^{n-1} q_m)^{k-1}$

$= O(1) \sum_{m=0}^{\infty} (Q_m Q_{m+1} q_m^{-1} q_{m+1}^{-1})^k q_m^{1-k} |\Delta q_m|^k |\Delta\varepsilon_m|^k \sum_{n=m+1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}$

$= O(1) \sum_{m=0}^{\infty} (Q_{m+1} q_m^{-1} q_{m+1}^{-1} \Delta q_m)^k (Q_m q_m^{-1})^{k-1} |\Delta\varepsilon_m|^k$

$= O(1)$ (using (iv) and (vi) **(4.18)**

A similar treatment gives us

$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{n2}^3|^k$

$= \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (q_n (Q_n Q_{n-1})^{-1})^k \left| \sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} B_{m-j} Q_{m+1} q_{m+1}^{-1} \Delta^2 \varepsilon_m \right|^k$

$\leq \sum_{n=1}^{\infty} (\varphi_n q_n Q_n^{-1})^{k-1} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \left( \sum_{j=0}^{n-1} R_j t_j \sum_{m=j}^{n-1} |B_{m-j}| Q_{m+1} q_{m+1}^{-1} |\Delta^2 \varepsilon_m| \right)^k$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \left( \sum_{m=0}^{n-1} Q_{m+1} q_{m+1}^{-1} |\Delta^2 \varepsilon_m| \sum_{j=0}^{m} R_j B_{m-j} \right)^k$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} (\sum_{m=0}^{n-1} Q_m Q_{m+1} q_{m+1}^{-1} |\Delta^2 \varepsilon_m|)^k$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k} \sum_{m=0}^{n-1} (Q_{m+1} q_{m+1}^{-1})^k Q_m^k q_m^{1-k} |\Delta^2 \varepsilon_m|^k \, (Q_{n-1}^{-1} \sum_{m=0}^{n-1} q_m)^{k-1}$

$= O(1) \sum_{m=0}^{\infty} (Q_{m+1} q_{m+1}^{-1})^k (Q_m q_m^{-1})^{k-1} Q_m \, |\Delta^2 \varepsilon_m|^k \sum_{n=m+1}^{\infty} \varphi_n^{\delta k} q_n Q_n^{-1} Q_{n-1}^{-k}$

$= O(1)$, (using (vi) and (v)) **(4.19)**

Now we use (iv) and (vi) to obtain

$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |U_{n2}^4|^k = \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (|\Delta \varepsilon_n| Q_{n-1}^{-1})^k |\sum_{m=0}^{n-1} R_m u_m B_{n-1-m}|^k$

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (|\Delta \varepsilon_n| Q_{n-1}^{-1})^k Q_{n-1}^k$

(using (i) and Lemma 2.3)

$= O(1) \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \, (|\Delta \varepsilon_n|)^k$

$= O(1) \sum_{n=1}^{\infty} (\varphi_n q_n Q_n)^{k-1} \left(\varphi_n^{\delta k/(k-1)} Q_n q_n^{-1}\right)^{k-1} |\Delta \varepsilon_n|^k$

$= O(1) \sum_{n=1}^{\infty} \left(\varphi_n^{\delta k/(k-1)} Q_n q_n^{-1}\right)^{k-1} |\Delta \varepsilon_n|^k$

$= O(1)$, (using (vii)) **(4.20)**

Now using equations (4.13) – (4.20) in (4.12), we find that the necessary and sufficient condition for $\sum a_n \varepsilon_n$ to be summable $\varphi - |\overline{N}, \ q_n, \delta|_k$ whenever $\sum a_n$ is |N, p, q| bounded is that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \, |U_{n3}|^k < \infty$$

i.e.

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} (q_n Q_n^{-1})^k \, |s_n|^k |\varepsilon_n|^k < \infty.$$

Which is the required condition (viii).

This completes the proof of the theorem.

## 5. Concluding Remarks:

By taking $\varphi_n = n$ and $\delta = 0$, we get Theorem 2.3 and in that case the condition (vi) will be obvious, while condition (vii) reduces to condition (iii). By selecting the values of $\delta$ and $\varphi_n$ appropriately, we can also get the results of Mazhar [5] and Sulaiman [6], and thus all the other results generalized by them.

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