RIGHT k-FIBONACCI SEQUENCE AND RELATED IDENTITIES

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ABSTRACT

Fibonacci sequence $\{F_n\}$ *is defined by the recurrence relation* $F_n = F_{n-1} + F_{n-2},\ n \geq 2$ *with initial condition* $F_0 = 0,\ F_1 = 1$ *.This sequence has been generalized in many ways, some by preserving the initial conditions, and other by preserving by recurrence relation. In this paper, we study the right k-Fibonacci sequence* $\left\{F_{k,n}^{R}\right\}$ *defined by the recurrence relation,* $F_{k,n}^{R} = F_{k,n-1}^{R} + kF_{k,n-2}^{R},\ n \geq 2$ *with initial conditions* $F_{k,0}^{R} = 0$ *and* $F_{k,1}^{R} = 1$ *We derive some interesting identities for this sequence.*

1. Introduction:

Fibonacci sequence $\{F_n\}$, named after Leonardo Pisano Fibonacci (1170−1250), is defined as $F_0 = 0,\ F_1 = 1$ and $F_n = F_{n-1} + F_{n-2},\ n \geq 2$ which gives the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 … . The Fibonacci numbers are perhaps most famous for appearing in the rabbit-breeding problem, introduced by Leonardo de Pisa in 1202 in his book called *Liber Abaci*. However, they also occur in Pascal's triangle [1].

Some authors ([2, 3, 4, 5, 6]) have generalized the Fibonacci sequence by preserving the recurrence relation and altering the first two terms of the sequence, while others ([7, 8, 9, 10, 11, 12, 13]) have generalized the Fibonacci sequence by preserving the first two terms of the sequence but altering the recurrence relation slightly. In [14, 15, 16] new generalization depends on two real parameters used in a non-linear recurrence relation.

In this paper, a new generalization of the Fibonacci numbers introduced. It should be noted that the recurrence formula of these numbers depends on one real parameter. These numbers extend the definition of the k-Fibonacci numbers given in [7, 8] where k was a positive integer. We now introduce a further generalization of Fibonacci sequence as the *right k-Fibonacci sequence* $\left\{F_{k,n}^{R}\right\}$ using recurrence relation on one real parameter k given by

$$F_{k,n}^{R} = F_{k,n-1}^{R} + kF_{k,n-2}^{R},\ n \geq 2 \text{ where } F_{k,0}^{R} = 0 \text{ and } F_{k,1}^{R} = 1.$$

Some of the terms of this sequence are shown in the following table:

n	$F_{k,n}^{R}$
0	0
1	1
2	1
3	$1+k$
4	$1+2k$
5	$1+3k+k^2$
6	$1+4k+3k^2$
7	$1+5k+6k^2+k^3$
8	$1+6k+10k^2+4k^3$
9	$1+7k+15k^2+10k^3+k^4$
10	$1+8k+21k^2+20k^3+5k^4$
11	$1+9k+28k^2+35k^3+15k^4+k^5$
12	$1+10k+36k^2+56k^3+35k^4+6k^5$

13	$1+11k+45k^2+84k^3+70k^4+21k^5+k^6$
14	$1+12k+55k^2+120k^3+126k^4+56k^5+7k^6$
15	$1+13k+66k^2+165k^3+210k^4+126k^5+28k^6+k^7$
16	$1+14k+78k^2+220k^3+330k^4+252k^5+84k^6+8k^7$

If $k=1$, this sequence is a classical Fibonacci sequence and for $k=2$, we get classical Pell's sequence. In this paper we obtain some interesting identities whose corresponding counterpart is well-known in Fibonacci sequence.

2. Some basic identities:

Lemma 2.1: $\gcd(F_{k,n}^R, F_{k,n+1}^R)=1,\ \forall\ n=0,1,2,3,\cdots$

Proof: Suppose that $F_{k,n}^R$ and $F_{k,n+1}^R$ are both divisible by a positive integer *d*. Then clearly

$$F_{k,n+1}^R - F_{k,n}^R = F_{k,n}^R + kF_{k,n-1}^R - F_{k,n}^R = kF_{k,n-1}^R$$

will also be divisible by *d*. Then right hand side of this result is divisible by *d*. This gives $d\,|\,F_{k,n-1}^R$.Continuing this argument we see that $d\,|\,F_{k,n-2}^R$, $d\,|\,F_{k,n-3}^R$ and so on. Eventually, we must have $d\,|\,F_{k,1}^R$. Since $F_{k,1}^R=1$ we get $d=1$, which proves the required result.

We now find an expression for the sum of first *n* terms of this sequence.

Lemma 2.2 $F_{k,1}^R + F_{k,2}^R + F_{k,3}^R + \cdots + F_{k,n}^R = \frac{1}{k}(F_{k,n+2}^R - 1)$.

Proof: We have $F_{k,n}^R = F_{k,n-1}^R + kF_{k,n-2}^R,\ \ n\ge 2$. Replacing n by 2, 3, 4,… we get

$$F_{k,2}^R = F_{k,1}^R + kF_{k,0}^R$$

$$F_{k,3}^R = F_{k,2}^R + kF_{k,1}^R$$

$$F_{k,4}^R = F_{k,3}^R + kF_{k,2}^R$$

$$\vdots$$

$$F_{k,n-2}^R = F_{k,n-3}^R + kF_{k,n-4}^R$$

$$F_{k,n-1}^{R} = F_{k,n-2}^{R} + kF_{k,n-3}^{R}$$

$$F_{k,n}^{R} = F_{k,n-1}^{R} + kF_{k,n-2}^{R}$$

Now adding all these equations term by term, we get

$$F_{k,2}^{R} + F_{k,3}^{R} + \cdots + F_{k,n}^{R} = (\mathrm{F}_{k,1}^{R} + \mathrm{F}_{k,2}^{R} + \cdots + \mathrm{F}_{k,\mathrm{n}-1}^{R}) + \mathrm{k}(\mathrm{F}_{k,0}^{R} + \mathrm{F}_{k,1}^{R} + \mathrm{F}_{k,2}^{R} + \cdots + \mathrm{F}_{k,\mathrm{n}-2}^{R})$$

$$\therefore \quad F_{k,n}^{R} - F_{k,1}^{R} = k(F_{k,0}^{R} + \mathrm{F}_{k,1}^{R} + \mathrm{F}_{k,2}^{R} + \cdots + \mathrm{F}_{k,n}^{R}) - k\,\mathrm{F}_{k,n-1}^{R} - \mathrm{k}\,F_{k,n}^{R}$$

$$\therefore \quad k(F_{k,1}^{R} + F_{k,2}^{R} + \cdots + F_{k,n}^{R}) = F_{k,n}^{R} + kF_{k,n-1}^{R} + kF_{k,n}^{R} - F_{k,1}^{R}$$

$$= F_{k,n+1}^{R} + kF_{k,n}^{R} - 1 = F_{k,n+2}^{R} - 1$$

$$\therefore \quad F_{k,1}^{R} + F_{k,2}^{R} + F_{k,3}^{R} + \cdots + F_{k,n}^{R} = \frac{1}{k}(F_{k,n+2}^{R} - 1)$$

Lemma 2.3 Sum of the first $2n$ right k - Fibonacci numbers is given by

$$F_{k,1}^{R} + F_{k,2}^{R} + F_{k,3}^{R} + \cdots + F_{k,2n}^{R} = \frac{1}{k}(F_{k,2n+2}^{R} - 1).$$

Proof: We have $F_{k,n}^{R} = F_{k,n-1}^{R} + kF_{k,n-2}^{R}, \;\; n \geq 2$. Replacing n by 2, 3, 4,… we get

$$F_{k,2}^{R} = F_{k,1}^{R} + kF_{k,0}^{R}$$

$$F_{k,3}^{R} = F_{k,2}^{R} + kF_{k,1}^{R}$$

$$F_{k,4}^{R} = F_{k,3}^{R} + kF_{k,2}^{R}$$

$$\vdots$$

$$F_{k,2n-2}^{R} = F_{k,2n-3}^{R} + kF_{k,2n-4}^{R}$$

$$F_{k,2n-1}^{R} = F_{k,2n-2}^{R} + kF_{k,2n-3}^{R}$$

$$F_{k,2n}^{R} = F_{k,2n-1}^{R} + kF_{k,2n-2}^{R}$$

Now adding all these equations term by term, we get

$$F_{k,2}^{R} + F_{k,3}^{R} + \cdots + F_{k,2n}^{R} = (1+\mathrm{k})(\mathrm{F}_{k,1}^{R} + \mathrm{F}_{k,2}^{R} + \cdots + \mathrm{F}_{k,2\mathrm{n}-2}^{R}) + \mathrm{F}_{k,2\mathrm{n}-1}^{R}$$

$$F_{k,1}^R + F_{k,2}^R + \cdots + F_{k,2n}^R = F_{k,1}^R + (1+k)(F_{k,1}^R + F_{k,2}^R + \cdots + F_{k,2n}^R) - (1+k)(F_{k,2n-1}^R + F_{k,2n}^R) + F_{k,2n-1}^R$$

$$\therefore \quad k(F_{k,1}^R + F_{k,2}^R + \cdots + F_{k,2n}^R) = F_{k,2n}^R + kF_{k,2n-1}^R + kF_{k,2n}^R - F_{k,1}^R$$

$$= F_{k,2n+1}^R + kF_{k,2n}^R - 1 = F_{k,2n+2}^R - 1$$

$$\therefore \quad F_{k,1}^R + F_{k,2}^R + F_{k,3}^R + \cdots + F_{k,2n}^R = \frac{1}{k}(F_{k,2n+2}^R - 1).$$

The following results follow immediately from above two lemmas.

Corollary 2.4 $F_{k,n+2}^R \equiv 1 (\mathrm{mod}\, m)$ and $F_{k,2n+2}^R \equiv 1 (\mathrm{mod}\, m)$

We next find the sum of first n right k −Fibonacci numbers with only odd or even subscripts.

Lemma 2.5 $F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R = \dfrac{1}{k(2-k)}(F_{k,2n+2}^R - kF_{k,2n+1}^R + k - 1)$.

Proof: We have $F_{k,n}^R = F_{k,n-1}^R + kF_{k,n-2}^R, \quad n \geq 2$. Replacing n by 3, 5, 7 … we get

$$F_{k,3}^R = F_{k,2}^R + kF_{k,1}^R$$

$$F_{k,5}^R = F_{k,4}^R + kF_{k,3}^R$$

$$F_{k,7}^R = F_{k,6}^R + kF_{k,5}^R$$

$$\vdots$$

$$F_{k,2n-1}^R = F_{k,2n-2}^R + kF_{k,2n-3}^R$$

Adding all these equations term by term and using Lemma: 2.3, we get

$$F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R = F_{k,1}^R + (F_{k,2}^R + F_{k,4}^R + \cdots + F_{k,2n-2}^R) + k(F_{k,1}^R + F_{k,3}^R + \cdots + F_{k,2n-3}^R)$$

$$\therefore\ 2(F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R) = 1 + (F_{k,1}^R + F_{k,2}^R + \cdots + F_{k,2n-1}^R + F_{k,2n}^R) + k(F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R) - F_{k,2n}^R - kF_{k,2n-1}^R$$

$$\therefore\ (2-k)(F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R) = 1 - (F_{k,2n}^R + kF_{k,2n-1}^R) + (F_{k,1}^R + F_{k,2}^R + \cdots + F_{k,2n}^R)$$

$$\therefore\ (2-\mathrm{k})(F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R) = 1 - F_{k,2n+1}^R + \frac{1}{k}(F_{k,2n+2}^R - 1)$$

$$\therefore\ F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R = \frac{1}{k(2-k)}(F_{k,2n+2}^R - kF_{k,2n+1}^R + k - 1)$$

Lemma 2.6 $F_{k,2}^R + F_{k,4}^R + F_{k,6}^R + \cdots + F_{k,2n}^R = \frac{1}{k(2-k)}(F_{k,2n+2}^R - k^2 F_{k,2n}^R - 1)$.

Proof: We have $F_{k,n}^R = F_{k,n-1}^R + kF_{k,n-2}^R,\quad n \geq 2$. Replacing *n* by 2, 4, 6, … we get

$$F_{k,2}^R = F_{k,1}^R + kF_{k,0}^R$$

$$F_{k,4}^R = F_{k,3}^R + kF_{k,2}^R$$

$$F_{k,6}^R = F_{k,5}^R + kF_{k,4}^R$$

$$\vdots$$

$$F_{k,2n}^R = F_{k,2n-1}^R + kF_{k,2n-2}^R$$

Adding all these equations term by term, we get

$$F_{k,2}^R + F_{k,4}^R + F_{k,6}^R + \cdots + F_{k,2n}^R = (F_{k,1}^R + F_{k,3}^R + F_{k,5}^R + \cdots + F_{k,2n-1}^R) + k(F_{k,2}^R + F_{k,4}^R + F_{k,6}^R + \cdots + F_{k,2n-2}^R)$$

$$\therefore\quad 2(F_{k,2}^R + F_{k,4}^R + F_{k,6}^R + \cdots + F_{k,2n}^R) = (F_{k,1}^R + F_{k,2}^R + F_{k,3}^R + \cdots + F_{k,2n}^R) + k(F_{k,2}^R + F_{k,4}^R + F_{k,6}^R + \cdots + F_{k,2n}^R) - kF_{k,2n}^R$$

$$\therefore \quad (2-k)(F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n}) = (F^R_{k,1}+F^R_{k,2}+F^R_{k,3}+\cdots+F^R_{k,2n}) - kF^R_{k,2n}$$

$$= \frac{1}{k}(F^R_{k,2n+2}-1) - kF^R_{k,2n}$$

$$\therefore \quad F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n} = \frac{1}{k(2-k)}(F^R_{k,2n+2} - k^2F^R_{k,2n} - 1)$$

Now multiplication of two consecutive generalized Fibonacci numbers is given by following lemma.

Lemma 2.7 $F^R_{k,n}\,F^R_{k,n+1} = F^{R\,2}_{k,n} + kF^{R\;2}_{k,n-1} + k^2F^{R\;2}_{k,n-2} + \cdots + k^{n-1}F^{R\,2}_{k,1} = \sum_{i=1}^{n} k^{n-i}F^{R\,2}_{k,i}$

Proof We have $F^R_{k,n} = F^R_{k,n-1} + kF^R_{k,n-2}$ and $F^R_{k,n+1} = F^R_{k,n} + kF^R_{k,n-1}$

$$F^R_{k,n}F^R_{k,n+1} = F^R_{k,n}(F^R_{k,n} + kF^R_{k,n-1}) = \mathrm{F}^{R\;2}_{k,n} + kF^R_{k,n-1}(F^R_{k,n-1} + kF^R_{k,n-2})$$

$$= \mathrm{F}^{R\;2}_{k,n} + kF^{R\;\;2}_{k,n-1} + \mathrm{k}^2\mathrm{F}^R_{k,n-2}(F^R_{k,n-2} + kF^R_{k,n-3})$$

$$= \mathrm{F}^{R\;2}_{k,n} + kF^{R\;\;2}_{k,n-1} + \mathrm{k}^2\mathrm{F}^{R\;\;2}_{k,n-2} + \mathrm{k}^3\mathrm{F}^R_{k,n-3}(F^R_{k,n-3} + kF^R_{k,n-4})$$

$$= \mathrm{F}^{R\;2}_{k,n} + kF^{R\;\;2}_{k,n-1} + \mathrm{k}^2\mathrm{F}^{R\;\;2}_{k,n-2} + \cdots + k^{n-1}F^R_{k,1}(\mathrm{F}^R_{k,1} + F^R_{k,0})$$

$$= \mathrm{F}^{R\;2}_{k,n} + kF^{R\;\;2}_{k,n-1} + \mathrm{k}^2\mathrm{F}^{R\;\;2}_{k,n-2} + \cdots + k^{n-1}F^{R\,2}_{k,1} = \sum_{i=1}^{n} k^{n-i}F^{R\,2}_{k,i}$$

3. Some more identities for right $k-$ Fibonacci numbers:

We now derive some more interesting identities for $F^R_{k,\mathrm{n}}$.

Lemma 3.1 $F^R_{k,\mathrm{m+n}} = kF^R_{k,\mathrm{m}-1}F^R_{k,n} + F^R_{k,\mathrm{m}}F^R_{k,n+1}$.

Proof: Let *m* be the fixed positive integer. We proceed by inducting on *n*.

For $n=1$, we have $F^R_{k,\mathrm{m+1}} = kF^R_{k,\mathrm{m}-1}F^R_{k,1} + F^R_{k,\mathrm{m}}F^R_{k,2}$.

Since $F^R_{k,1} = F^R_{k,2} = 1$, we have $F^R_{k,m+1} = F^R_{k,m} + kF^R_{k,m-1}$, which is true. This proves the result for $n=1$.

Now let us assume that the result is true for all integers up to some integer '*t*'.

Thus both $F^R_{k,\mathrm{m}+t} = kF^R_{k,\mathrm{m}-1}F^R_{k,t} + F^R_{k,\mathrm{m}}F^R_{k,t+1}$ and $F^R_{k,\mathrm{m}+(t-1)} = kF^R_{k,\mathrm{m}-1}F^R_{k,t-1} + F^R_{k,\mathrm{m}}F^R_{k,t}$ holds.

Now, from these two results we have

$$F^R_{k,\mathrm{m}+t} + kF^R_{k,m+(t-1)} = kF^R_{k,\mathrm{m}-1}(F^R_{k,t} + kF^R_{k,t-1}) + F^R_{k,m}(F^R_{k,t+1} + kF^R_{k,t})$$

$$= kF^R_{k,\mathrm{m}-1}F^R_{k,t+1} + F^R_{k,\mathrm{m}}F^R_{k,t+2}$$

$$= kF^R_{k,\mathrm{m}-1}F^R_{k,t+1} + F^R_{k,\mathrm{m}}F^R_{k,(t+1)+1} = F^R_{k,\mathrm{m}+(t+1)}.$$

which is obviously true. Thus by the principal of mathematical induction, the result is true for all positive integers *n*.

It is often useful to extend the sequence of right k- Fibonacci numbers backward with negative subscripts. In fact if we try to extend the right k- Fibonacci sequence back wards still keeping to the same rule, we get the following:

n	$F^R_{k,\mathrm{n}}$
-1	$\frac{1}{k}$
-2	$-\frac{1}{k^2}$
-3	$\frac{1+k}{k^3}$
-4	$-\frac{1+2k}{k^4}$
-5	$\frac{1+3k+k^2}{k^5}$
-6	$-\frac{1+4\mathrm{k}+3\mathrm{k}^2}{k^6}$
-7	$\frac{1+5k+6k^2+k^3}{k^7}$

-8	$-\frac{1+6k+10k^2+4k^3}{k^8}$

Thus the sequence of right $k-$ Fibonacci numbers is a bilateral sequence, since it can be extended infinitely in both directions. From this table and from the table of values of $F_{k,n}^R$, the following result follows immediately.

Lemma 3.2: $F_{k,-n}^R = \frac{(-1)^{n+1}}{k^n} F_{k,n}^R,\ n \geq 1$.

We now obtain the extended d'Ocagne's Identity for this sequence.

Lemma 3.3: $F_{k,m-n}^R = \frac{(-1)^n}{k^n}\left(F_{k,m}^R F_{k,n+1}^R - F_{k,m+1}^R F_{k,n}^R\right)$.

Proof: Replacing n by $-n$ in Lemma 3.1, we get

$$F_{k,m-n}^R = kF_{k,m-1}^R F_{k,-n}^R + F_{k,m}^R F_{k,-n+1}^R.$$

Using the definition of left $k-$ Fibonacci sequence and Lemma 3.2, we get

$$F_{k,m-n}^R = kF_{k,m-1}^R \frac{(-1)^{n+1}}{k^n} F_{k,n}^R + F_{k,m}^R \frac{(-1)^n}{k^{n-1}} F_{k,n-1}^R$$

$$= \frac{(-1)^n}{k^{n-1}}(F_{k,m}^R F_{k,n-1}^R - F_{k,m-1}^R F_{k,n}^R)$$

$$= \frac{(-1)^n}{k^{n-1}}[F_{k,m}^R \frac{1}{k}(F_{k,n+1}^R - F_{k,n}^R) - \frac{1}{k}(F_{k,m+1}^R - F_{k,m}^R)F_{k,n}^R]$$

$$\therefore\quad F_{k,m-n}^R = \frac{(-1)^n}{k^n}\left(F_{k,m}^R F_{k,n+1}^R - F_{k,m+1}^R F_{k,n}^R\right).$$

We next prove the divisibility property for $F_{k,n}^R$.

Lemma 3.4: $F_{k,m}^R \mid F_{k,mn}^R$; for any non-zero integers m and n.

Proof: Let m be any fixed positive integer. We proceed by inducting on n.

For $n = 1$, we have $F_{k,m}^R \mid F_{k,m}^R$, which is obvious. This proves the result for $n = 1$.

Now suppose the result is true for all integers n up to some integer 't'. i.e. we assume that $F^R_{k,m} \mid F^R_{k,mt}$ Then $F^R_{k,m(t+1)} = F^R_{k,mt+m} = kF^R_{k,mt-1}F^R_{k,m} + F^R_{k,mt}F^R_{k,m+1}$.

But by assumption, we have $F^R_{k,m} \mid F^R_{k,m\,t}$. Thus $F^R_{k,m}$ divides the entire right side of the above equation. Hence $F^R_{k,m} \mid F^R_{k,m\,(t+1)}$, which proves the result for all positive integers n.

Note: By Lemma 3.2 it is obvious that the above divisibility criterion holds for negative values of n also.

Lemma 3.5 ${F^R_{k,n}}^2 + \frac{1}{k}{F^R_{k,n+1}}^2 = \frac{1}{k}F^R_{k,2n+1}$.

Proof: Here also we use the principal of mathematical induction.

For $n=1$, we have ${F^R_{k,1}}^2 + \frac{1}{k}{F^R_{k,2}}^2 = 1 + \frac{1}{k} = \frac{1}{k}(1+\mathrm{k}) = \frac{1}{k}\mathrm{F}^R_{k,3}$.

This proves the result for $n=1$.

We assume that it is true for all integers up to some positive integer 't'.

$\therefore\quad {F^R_{k,t}}^2 + \frac{1}{k}{F^R_{k,t+1}}^2 = \frac{1}{k}\mathrm{F}^R_{k,2t+1}$ holds by assumption.

Now ${F^R_{k,t+1}}^2 + \frac{1}{k}{F^R_{k,t+2}}^2 = {\mathrm{F}^R_{k,t+1}}^2 + \frac{1}{k}(\mathrm{F}^R_{k,t+1} + k\,\mathrm{F}^R_{k,t})^2$

$$= {F^R_{k,t+1}}^2 + \frac{1}{k}({\mathrm{F}^R_{k,t+1}}^2 + 2k\,\mathrm{F}^R_{k,t}\mathrm{F}^R_{k,t+1} + k^2{F^R_{k,t}}^2)$$

$$= {F^R_{k,t+1}}^2 + k{F^R_{k,\mathrm{t}}}^2 + \frac{1}{k}({\mathrm{F}^R_{k,t+1}}^2 + k\,\mathrm{F}^R_{k,t}\mathrm{F}^R_{k,t+1} + kF^R_{k,t}\,\mathrm{F}^R_{k,t+1})$$

$$= k({\mathrm{F}^R_{k,t}}^2 + \frac{1}{k}{F^R_{k,t+1}}^2) + \frac{1}{k}[F^R_{k,t+1}(F^R_{k,t+1} + kF^R_{k,t}) + kF^R_{k,t}F^R_{k,t+1}]$$

$$= k(\frac{1}{k}F^R_{k,2t+1}) + \frac{1}{k}[\mathrm{F}^L_{k,t+1}\mathrm{F}^L_{k,t+2} + kF^L_{k,t}F^L_{k,t+1}]$$

$$= \mathrm{F}^R_{k,2t+1} + \frac{1}{k}(\mathrm{k}\,F^R_{k,t}\,\mathrm{F}^R_{k,t+1} + F^R_{k,t+1}F^R_{k,t+2})$$

$$= \mathrm{F}^R_{k,2t+1} + \frac{1}{k}F^R_{k,t+1+t+1} = \frac{1}{k}F^R_{k,2t+3} = \mathrm{F}^R_{k,2(t+1)+1}$$

This proves the result by induction.

Now, we derive a result which connects three consecutive right k- Fibonacci numbers with odd subscript.

Lemma 3.6 $F^R_{k,2n+5} - (2k+1)F^R_{k,2n+3} + k^2 F^R_{k,2n+1} = 0$

Proof By definition

$$F^R_{k,2n+5} = F^R_{k,2n+4} + kF^R_{k,2n+3} = (F^R_{k,2n+3} + kF^R_{k,2n+2}) + kF^R_{k,2n+3}$$

$$= (k+1)F^R_{k,2n+3} + kF^R_{k,2n+2}.$$

Now $F^R_{k,2n+5} - (2k+1)F^R_{k,2n+3} + k^2 F^R_{k,2n+1}$

$$= (k+1)F^R_{k,2n+3} + kF^R_{k,2n+2} - (2k+1)F^R_{k,2n+3} + k^2 F^R_{k,2n+1}$$

$$= (k+1)F^R_{k,2n+3} - (2k+1)F^R_{k,2n+3} + k(F^R_{k,2n+2} + kF^R_{k,2n+1})$$

$$= (k+1)F^R_{k,2n+3} - (2k+1)F^R_{k,2n+3} + kF^R_{k,2n+3} = 0$$

We finally prove the analogous of one of the oldest identities involving the Fibonacci numbers - Cassini's identity, which was discovered in 1680 by a French astronomer Jean – Dominique Cassini.

Lemma 3.7 $F^R_{k,n+1}.F^R_{k,n-1} - F^{R\,2}_{k,n} = (-1)^n.k^{n-1}$.

Proof : We have $F^R_{k,n+1}.F^R_{k,n-1} - F^{R\,2}_{k,n} = (F^R_{k,n} + kF^R_{k,n-1})F^R_{k,n-1} - F^{R\,2}_{k,n}$

$$= F^R_{k,n}F^R_{k,n-1} - F^{R\,2}_{k,n} + kF^{R\,2}_{k,n-1}$$

$$= F^R_{k,n}(F^R_{k,n-1} - F^R_{k,n}) + kF^{R\,2}_{k,n-1}$$

$$= F^R_{k,n}(-kF^R_{k,n-2}) + kF^{R\,2}_{k,n-1}$$

$$= -k(F^R_{k,n}F^R_{k,n-2} - F^{R\,2}_{k,n-1})$$

Repeating the same process successively for right side, we get

$$\therefore \; F^R_{k,n+1}F^R_{k,n-1} - F^{R\,2}_{k,n} = (-k)^2(F^R_{k,n-1}F^R_{k,n-3} - F^{R\,2}_{k,n-2})$$

$$= (-k)^n(F^R_{k,1}.F^R_{k,-1} - F^{R\,2}_{k,0})$$

$$= (-k)^n (1.\frac{1}{k} - 0) = (-1)^n .k^{n-1}$$

Since the value of $F_{k,1}^R = 1, F_{k,0}^R = 0, F_{k,-1}^R = \frac{1}{k}$, we have

$$\therefore \quad F_{k,n+1}^R .F_{k,n-1}^R - F_{k,n}^{R\ 2} = (-1)^n .k^{n-1}$$

Lemma 3.8 $F_{k,n}^R - c^{n-1} = (1-c)F_{k,n-1}^R + [(1-\mathrm{c})\,\mathrm{c} + k][\mathrm{F}_{k,0}^R\, c^{n-2} + \mathrm{F}_{k,1}^R\, c^{n-3} + \cdots + \mathrm{F}_{k,\mathrm{n}-2}^R]$

where $c = 1$

Proof We prove this result by the principal of mathematical induction.

For $n = 2$, we have $F_{k,2}^R - c = (1-c)\,\mathrm{F}_{k,1}^R + [(1-c)c + k]F_{k,0}^R$ which gives $1 - c = 1 - c$.

This proves the result for $n = 2$.

We assume that it is true for all integers up to some positive integer '*t*'.

$$F_{k,\mathrm{t}}^R - c^{t-1} = (1-c)F_{k,\mathrm{t}-1}^R + [(1-\mathrm{c})\,\mathrm{c} + k][\mathrm{F}_{k,0}^R\, c^{t-2} + \mathrm{F}_{k,1}^R\, c^{t-3} + \cdots + \mathrm{F}_{k,\mathrm{t}-2}^R],\ c = 1$$

To prove the result is true for $n = t + 1$.

Now $RHS = (1-c)F_{k,\mathrm{t}}^R + [(1-\mathrm{c})\,\mathrm{c} + k][\mathrm{F}_{k,0}^R\, c^{t-1} + \mathrm{F}_{k,1}^R\, c^{t-2} + \cdots + \mathrm{F}_{k,\mathrm{t}-1}^R]$

$$= (1-c)F_{k,\mathrm{t}}^R + [(\mathrm{k}-\mathrm{c})\,\mathrm{c}+1][\mathrm{F}_{k,\mathrm{t}-1}^R + c\,\mathrm{F}_{k,\mathrm{t}-2}^R + c^2\,\mathrm{F}_{k,\mathrm{t}-3}^R + \cdots + c^{t-2}\,\mathrm{F}_{k,1}^R + c^{t-1}F_{k,0}^R]$$

$$\begin{aligned} &= (F_{k,t}^R + kF_{k,t-1}^R) - \mathrm{c}\mathrm{F}_{k,t}^R + k(cF_{k,t-2}^R + c^2 F_{k,\mathrm{t}-3}^R + \cdots + c^{t-1}F_{k,0}^R) \\ &\quad + c(F_{k,\mathrm{t}-1}^R + cF_{k,\mathrm{t}-2}^R + \cdots + c^{t-2}F_{k,1}^R + \mathrm{c}^{t-1}\mathrm{F}_{k,0}^R) \\ &\quad - c^2 F_{k,\mathrm{t}-1}^R - c^3 F_{k,\mathrm{t}-2}^R - \cdots - c^{t-1}F_{k,1}^R - \mathrm{c}^t\mathrm{F}_{k,1}^R \end{aligned}$$

$$\begin{aligned} &= F_{k,t+1}^R - cF_{k,t}^R + c(F_{k,t-1}^R + kF_{k,t-2}^R) + c^2(F_{k,t-2}^R + kF_{k,t-3}^R) + \cdots + c^{t-1}(F_{k,1}^R + kF_{k,0}^R) \\ &\quad - \mathrm{c}^2\, F_{k,t-1}^R - \mathrm{c}^3\, F_{k,t-2}^R - \cdots - c^{t-1}F_{k,2}^R - c^k = F_{k,t+1}^R - c^t . \end{aligned}$$

$$\therefore F_{k,t+1}^R - c^t = (1-c)F_{k,\mathrm{t}}^R + [(1-\mathrm{c})\,\mathrm{c} + k][\mathrm{F}_{k,0}^R\, c^{t-1} + \mathrm{F}_{k,1}^R\, c^{t-2} + \cdots + \mathrm{F}_{k,\mathrm{t}-1}^R]$$

The result is true for $n = t+1$. This proves the result by induction.

Note: If we take $c = 1$ in this result, we have $F_{k,n}^{R} = 1 + k\sum_{i=1}^{n-2} F_{k,i}^{R}$.

Robinson [17] used the matrices to discover facts about the Fibonacci sequence. We now demonstrate a close link between matrices and right k- Fibonacci numbers. We define an important 2×2 matrices as follow, which plays a significant role in discussions concerning right k- Fibonacci sequence.

Lemma 3.9 If $U = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}$ then $U^n = \begin{bmatrix} kF_{k,n-1}^{R} & kF_{k,n}^{R} \\ F_{k,n}^{R} & F_{k,n+1}^{R} \end{bmatrix}$

Proof: We will prove this result by using principal of mathematical induction,

for $n = 1$, we have $U = \begin{bmatrix} kF_{k,0}^{R} & kF_{k,1}^{R} \\ F_{k,1}^{R} & F_{k,2}^{R} \end{bmatrix} = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}$

The result is proved for $n = 1$.

Suppose it is true for $n = t$. i.e. $U^t = \begin{bmatrix} kF_{k,t-1}^{R} & kF_{k,t}^{R} \\ F_{k,t}^{R} & F_{k,t+1}^{R} \end{bmatrix}$

Now $U^{t+1} = U^t U = \begin{bmatrix} kF_{k,t-1}^{R} & kF_{k,t}^{R} \\ F_{k,t}^{R} & F_{k,t+1}^{R} \end{bmatrix} \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} kF_{k,t}^{R} & kF_{k,t}^{R} + k^2 F_{k,t-1}^{R} \\ F_{k,t+1}^{R} & F_{k,t+1}^{R} + kF_{k,t}^{R} \end{bmatrix}$

$$= \begin{bmatrix} kF_{k,t}^{R} & kF_{k,t+1}^{R} \\ F_{k,t+1}^{R} & F_{k,t+2}^{R} \end{bmatrix}$$

Thus the result is true for $n = t+1$. This proves the result by induction.

Lemma: 3.8 can be prove by using matrix, We have, $U = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}$ then $|U| = -k$

$$\text{also } U^n = \begin{bmatrix} kF^R_{k,n-1} & kF^R_{k,n} \\ F^R_{k,n} & F^R_{k,n+1} \end{bmatrix} \text{ then } \left|U^n\right| = kF^R_{k,n-1}F^R_{k,n+1} - kF^{R\,2}_{k,n}$$

$$\therefore \left|U^n\right| = k\left[F^R_{k,n-1}F^R_{k,n+1} - F^{R\,2}_{k,n}\right] \qquad \therefore (-k)^n = k\left[F^R_{k,n-1}F^R_{k,n+1} - F^{R\,2}_{k,n}\right]$$

$$\therefore F^R_{k,n-1}F^R_{k,n+1} - F^{R\,2}_{k,n} = (-1)^n k^{n-1}.$$

Acknowledgement: This work has been supported in part by UGC Minor Project number F. 47-901/14 (WRO) from Ministry of Human resource Development, Govt. of India.

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