CHARACTERIZATIONS OF INFINITE MATRICES ON SOME PARANORMED β −SPACES

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ABSTRACT

This paper uses the definition of a paranormed β −space to determine the necessary and sufficient conditions for a sequence $((A_n(x))$ of continuous linear functionals to be in the spaces $l_\infty(q)$ and $c_0(q)$ for each x belonging to a paranormed β −space. It is observed that the work fills a gap in the existing literature.

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Key Words: Paranorm, Paranormed β −spaces, Matrix transformation, Kőthe-Toeplitz dual.

1. Introduction

A paranormed space (X, g) whose topology is generated by a paranorm g is a topological linear space, where g is a real subadditive function on X which satisfies $(\theta) = 0$, $g(x) \geq 0$, $g(-x) = g(x)$, $g(x + y) \leq g(x) + g(y)$, $\forall x, y \in X$, and such that multiplication is continuous. θ is the zero sequence in X and by continuity of multiplication we mean if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$, then $g(\lambda_n x_n - \lambda x) \to 0$. A paranorm is said to be total if $g(x) = 0$ implies $x = \theta$.

A paranormed β −space is defined in Maddox [1] and is captured as follows: Let (X_n) be a sequence of subsets of X such that $\theta \in X_1$ and such that if $x, y \in X_n$ then $x \pm y \in X_{n+1}$ $\forall n \in N$; then (X_n) is called an α −space in X. If $X = \cup_{n=1}^{\infty} X_n$, where (X_n) is an α −sequence in X and each X_n is nowhere dense in X, then X is called an α −space.

Otherwise X is called a β −space. It is clear then that every α −space is of the first category, whence we see that any complete paranormed space is a β −space. This definition is a generalization of the definition of Sargent [2].

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n, k = 1, 2, \ldots)$ and X, Y be two nonempty subsets of the space ω of all complex sequences. The matrix A is said to define a matrix transformation from X into Y and written $A : X \to Y$ if for every $x = (x_k) \in X$ and every integer n we have

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

If the sequence $Ax = \big(A_n(x)\big)$ exists, then it is called the transformation of x by the matrix A. Further, $A \in (X, Y)$ if and only if $Ax \in Y$, whenever $x \in X$; where the pair (X, Y) denotes the class of matrices A. For different sets of spaces X and Y, the necessary and sufficient conditions have been established for a sequence A to be in the class (X, Y).

For a sequence of positive numbers $p_k < 1$ $(k = 1, 2, \ldots)$, denoting by $l(p_k)$ the totality of $x = (x_1, x_2, \ldots)$ for which $\sum_{k=1}^{\infty} | x_k |^{p_k} < +\infty$ was first considered by Halperin and Nakano [3]. Simons [4] considered the case $0 < p_k \leq 1$ and defined

$$l(p_k) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} | x_k |^{p_k} < \infty\}$$

where ω is the set of all real or complex sequences (x_k); and proved that the set $l(p_k)$ is a linear space under the coordinate-wise definitions of addition and scalar multiplication. He further proved that it is complete linear topological space.

For the case $\sup p_k < \infty$, it was shown in Maddox [5] that $l(p_k)$ is a linear space and further shown to be a paranormed sequence space in the most general case when $p_k = O(1)$, see [6].

For $p_k > 0$ the space $l(p, s)$ is defined by

$$l(p, s) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} k^{-s} | x_k |^{p_k} < \infty, s \geq 0\},$$

It was also mentioned that this space is paranormed by

$$g(x) = (\sum_k k^{-s} | x_k |^{p_k})^{1/M}$$

whenever $p \in l_\infty$ by Bulut-Çakar [7]; and they further observed that

$$l(p_k) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} | x_k |^{p_k} < \infty, \ p_k > 0\}$$

is a special case of $l(p, s)$ corresponding to $s = 0$ and that $l(p, s) \supset l(p)$.

Our sequence space of interest namely:

$$l^v(p, t) = \{x = (x_x) \in \omega : \sum_{k=1}^{\infty} k^{-t} | x_k v_k|^{p_k} < \infty, \ t \geq 0\}$$

for bounded sequence $p = (p_k)$ of strictly positive numbers and $v = (v_k)$ any fixed sequence of non-zero complex numbers such that $\lim_{k\to\infty} \inf | v_k |^{\frac{1}{k}} = \rho \ (0 < \rho < \infty)$, is defined in Bilgin [8]. When $t = 0, v_k = 1$ and $p_k = 1$ for every $k \in N$, the space $l^v(p, t)$ becomes the space $l(p_k)$ of Maddox [5].

The space $l^v(p, t)$ is complete in its topology paranormed by

$$g(x) = \left(\sum_{k=1}^{\infty} k^{-t} | x_k v_k|^{p_k} \right)^{1/M}$$

where $M = \max(1, H)$ with $H = \sup_k p_k$. Thus, it is also a BK $-$space, since, it is well known that every complete paranormed space is BK. The space has $(e^{(k)})$ as basis, where $e^{(k)}$ is a sequence with 1 in the kth place and zero elsewhere.

Let E be a nonempty subset of ω. Then E^+ denotes the generalized Kőthe-Toeplitz dual of E defined by

$$E^+ = \{(a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for every } x \in E\}$$

It was further observed in Lascarides [9] that Kőthe-Toeplitz duality possesses the following features:

(i) E^+ is a linear subspace of ω for every $E \subset \omega$.

(ii) $E \subset Y$ implies $E^+ \supset Y^+$ for every $E, Y \subset \omega$.

(iii) $E^{++} = (E^+)^+ \supset E$ for every $E \subset \omega$.

(iv) $(\cup E_i)^+ = \cap E_i^+$ for every family $\{ E_i\}$ with $E_i \subset \omega$ and $i \in N$.

Any subset E of ω is perfect or ω is perfect or Kőthe-Toeplitz reflexive if and only if $E^{++} = E$. For instance E^+ is perfect for every E; and that if E is perfect then it is a linear space. Further, if $E \subset \omega$, and E is a Kőthe space, the E solid; and if E is solid then $E^{\alpha} = E^{\beta} = E^{\gamma}$ are the $\alpha-$, $\beta-$ and $\gamma-$ duals of E, respectively. That E is solid or total means when $x \in E$ and $|y_k| \leq |x_k|$, $\forall k \in N$ together imply $y \in E$, (see Maddox [10].

Let $E(p)$ denote the set $l^{v}(p,t)$, we state its generalized Kőthe-Toeplitz dual $E^{+}(p)$ as well as its continuous dual. To do this, we need some working lemmas. So, let $q = (q_n)$ denote a sequence of strictly positive real numbers. If q is bounded with $H = \max(\sup q_n, 1)$ then $c_0(q,t) = c_0(H/q,t)$, $l_\infty(q,t) = l_\infty(H/q,t)$ and $c(q,t) = c(H/q,t)$, (see Maddox [6]).

Lemma 1(Theorem 1 [11]**)**: Let X be a paranormed space and let (A_n) be a sequence of elements of X^*, and suppose also that q is bounded. Then

(i) $\sup_n(||A_n||_M)^{q_n} < \infty$ for some $M > 1$ implies

(ii) $(A_n(x)) \in l_\infty(q)$ for every $x \in X$,

and the converse is truee if X is a $\beta-$space.

Lemma 2 (Theorem 2, [11]**)**: Let X be a paranormed space and let (A_n) be a sequence of elements of X^*.

(1) If X has a fundamental set G and if q is bounded, then the following propositions

(iii) $(A_n(b)) \in c_0(q)$ for any $b \in G$,

(iv) $\lim_M \limsup_n(||A_n||_M)^{q_n} = 0$,

together imply

(v) $\left(A_n(x)\right) \in c_0(q)$ for every $x \in X$.

(2) If $q_n \to 0$ $(n \to \infty)$ then (iv) implies (v)

(3) Let X be a $\beta-$space, then (v) implies (iv) even if q is unbounded.

Lemma 3 (Theorem 3 [7]):

(i) If $1 < p_k < sup_k p_k = H < \infty$, fot each $k \in N$ then $A \in (l(p,s), l_\infty)$ if and only if there exists an integer D > 1 such that

$$\sup_n \sum_{k=1}^{\infty} | a_{nk} |^{q_k} M^{-q_k} k^{t(q_k-1)} < \infty \qquad (1)$$

(ii) If $0 < m = inf_k p_k \leq p_k \leq 1$ for each$k \in N$, then

$$A \in (l(p,t), l_\infty) \Leftrightarrow K = sup_{n,k}| a_{nk} |^{p_k} k^t < \infty \qquad (2)$$

Lemma 4 (Theorem 4 [7]**):**

(i) If $1 < p_k < sup_k p_k = H < \infty$, fot each $k \in N$ then $A \in (l(p,s), c)$ if and only if together with (1) the condition

$$a_{nk} \to 0 \quad (n \to \infty,\ k \text{ fixed}) \qquad (3)$$

holds.

(ii) If $0 < m = inf_k p_k \leq p_k \leq 1$ for each$k \in N$, then $A \in (l(p,s),\ c)$ if and only if the conditions (2) and (3) hold.

Lemma 5 (Lemma 2.2 [8]**)**

(a) If $1 < p_k \leq \sup p_k < \infty$ and ${p_k}^{-1} + {q_k}^{-1} = 1$, $k = 0, 1, 2, \ldots$, then

$(l^v(p,t)^\alpha = M^v(p,t)$ and $(l^v(p,t)^*$ is isomorphic to $M^v(p,t)$, where

$M^v(p,t) = \{a = (a_k): \sum_{k=1}^{\infty} | a_k v_k^{-1} |^{s_k} \cdot k^{t(q_k-1)} \cdot N^{-s_k/p_k} < \infty,\ t \geq 0\}$.

(b) If $0 < \inf p_k \leq p_k < 1$, then $(l^v(p,t)^\beta = l_\infty^v(p,t)$ and $(l^v(p,t)^*$ is isomorphic to $M^v(p,t)$, where

$$l_\infty^v(p,t) = \{a = (a_k) : sup_k| a_k v_k^{-1} |^{p_k} k^t < \infty,\ t \geq 0\}$$

Lemma 6 (Theorem 2 [7]:

(i) If $1 < p_k < sup_k p_k = H < \infty$, fot each $k \in N$ then $l^*(p,s)$, i.e. the continuous dual of $l(p,s)$ is isomorphic to E (p, s), which is defined as

$$E(p,s) = \left\{a = (a_k): \sum_{k=1}^{\infty} k^{s(q_k-1)} N^{-q_k/p_k} [a_k]^{q_k}, s \geq 0, for\ some\ integer\ N > 1\right\}$$

(ii) If $0 < m = inf_k p_k \leq p_k \leq 1$ for each$k \in N$, then $l^*(p,s)$ is isomorphic to m(p,s), which is defined as

$$\mathrm{m}(p,s) = \{a = (a_k)\}: sup_k k^s\ |a_k|^{p_k}, \infty, s \geq 1 \qquad (4)$$

2. Main Results

Let $p = (p_k)$ be a sequence of strictly positive real numbers, define $s = (s_k)$ as $p_k^{-1} + s_k^{-1} = 1$, $\forall k$ then for p and $v = (v_k)$ of any fixed sequence of non-zero complex numbers, we shall prove the following two results to characterize the classes $(l^v(p,t): l_\infty(q))$ and $(l^v(p,t): c_0(q))$ for both the cases $1 < p_k < \infty$ and $0 < p_k \leq 1$.

Theorem A:

(i) Let $0 < p_k \leq 1$, $p_k^{-1} + s_k^{-1} = 1$ for every k, and let q be bounded. Then $A \in (l^v(p,t): l_\infty(q))$ if and only if

$$\sup_n\{\sup_k k^{t/p_k}|\ a_{nk}v^{-1}|M^{-1/p_k}\}^{q_n} < \infty, \text{for some } M > 1. \qquad (5)$$

(ii) Suppose $1 < p_k \leq \sup p_k = H < \infty$, $p_k^{-1} + s_k^{-1} = 1$ for each $k \in N$; and let q be bounded. Then, $A \in (l^v(p,t): l_\infty(q))$, if and only if

$$T(B) = \sup_n \textstyle\sum_k k^{t(s_k-1)}B^{-s_k/q_n}(|a_{nk}||v_k^{-1}|)^{s_k} < \infty, \text{for some } B > 1. \qquad (6)$$

Proof: (i) Let $A \in (l^v(p,t): l_\infty(q))$. Then for each n, $(a_{n1}, a_{n2}, \dots) \in (l^v(p,t)^\alpha = l_\infty^v(p,t)$. Also, by lemma 5 $A_n \in (l^v(p,t)^*$ for each $n \in N$. We show that $||A_n||_M = \sup_k|\ a_{nk}v^{-1}|k^{t/p_k}M^{-1/p_k}$, for all $M > 1$ such that $||A_n||_M$ is defined. To do this, choose any $n \in N$. Now, if M is such that, for some sequence $(k_{(i)})$ of integers $|a_{nk(i)}v_{k(i)}^{-1}|M^{-1/p_{k(i)}} \geq i$ for each $i \in N$, then by defining

$$x^{(k(i))} = (M^{-1/p_{k(i)}}\mathrm{sgn}\big(a_{nk(i)} \cdot v_{k(i)}^{-1}\big))e^{(k(i))}, i = 1, 2, \dots$$

it follows that $||A_n||_M$ is defined. Since $(a_{n1}, a_{n2}, \dots) \in (l^v(p,t)^\alpha$ there is an integer $M_n \geq 1$ such that

$$|\ a_{nk}v_k^{-1}\ |^{p_k} \leq M_n, \quad \forall k.$$

Now choose $M \geq M_n$. Using $g(x) = \sum_k k^{-t}|\ x_k v_k\ |^{p_k} \leq 1/M$, since $M^{1/p_k}k^{-t/p_k}|\ x_k v_k^{-1}| \leq 1$, $\forall k$ and since $\sup_k p_k \leq 1$, we have:

$$|A_n(x)| \leq \textstyle\sum_k|\ a_{nk}x_k v_k^{-1}|$$

$$= \sum_k |(a_{nk} v_k^{-1}) x_k|$$

$$\leq \sum_k k^{-t/p_k} k^{t/p_k} |x_k| M^{1/p_k} M^{-1/p_k} |a_{nk} v_k^{-1}|$$

$$\leq \sum_k k^{-t} k^{t/p_k} |x_k|^{p_k} M \cdot M^{-1/p_k} |a_{nk} v_k^{-1}|$$

$$\leq \sup_k (k^{t/p_k} M^{-1/p_k} |a_{nk} v_k^{-1}|)$$

$$\leq M \cdot g(x) \sup_k (M^{-1/p_k} |a_{nk} v_k^{-1}|),$$

whence

$$||A_n||_M \leq \sup_k M^{-1/p_k} |a_{nk} v_k^{-1}|.$$

Given $\varepsilon > 0$ we can choose an $m > k$ such that

$$M^{-1/p_k} |a_{nk} v_k^{-1}| > \sup_k M^{-1/p_k} |a_{nk} v_k^{-1}| - \varepsilon.$$

Define $x = (M^{-1/p_k} sgn |a_{nk} v_k^{-1}| e^{(m)})$. Then we have $g(x) \leq 1/M$; and

$$A_n(x) > \sup_k M^{-1/p_k} |a_{nk} v_k^{-1}| < \varepsilon.$$

whence,

$$||A_n||_M = \sup_k M^{-1/p_k} |a_{nk} v_k^{-1}|.$$

Since $l^v(p,t)$ is complete paranormed space, by Lemma 2 it is a β −space; and thus by Lemma 1we must have (5) holding.

Conversely, let (5) hold. Then again it follows that for each n, $A_n \in (l^v(p,t))^*$ with $||A_n||_M = \sup_k |a_{nk} v_k^{-1}| M^{-1/p_k} k^{t/p_k},\ M$ such that $||A_n||_M$ is defined. And using Lemma 1 we must have that the sequence $(A_n(x)) \in l_\infty(q)$.

(ii) For each $n \in N$, define A_n by

$$A_n(x) = \sum_k a_{nk} x_k,$$

For sufficiency, let (6) hold. Then if $x \in l^v(p,t)$ we have for each n, assuming $q_n \leq 1,\ \forall n$,

$$|A_n(x)|^{q_n} \leq (\sum_k | a_{nk} x_k v_k^{-1}|)^{q_n}$$

$$= (\sum_k k^{t/p_k} k^{-t/p_k} |x_k| B^{-1/s_k} B^{-1/s_k} | a_{nk} v_k^{-1}|)^{q_n}$$

$$\leq (\sum_k k^{t(s_k-1)}|\, a_{nk} v_k^{-1}|^{s_k}\, B^{-\, s_k/q_n} + \sum_k\, B^{p_k/q_n} k^{-t} \cdot |x_k|^{p_k})^{q_n}$$

$$\leq (T(B))^{q_n} + B^H (g^H(x)^{q_n}$$

$$\leq T(B) + 1 + B^H (g^H(x) - 1)$$

which implies $A \in \left(l^v(p,t) \colon l_\infty(q)\right)$.

For necessity, let $A \in \left(l^v(p,t) \colon l_\infty(q)\right)$. Then $(a_{n1}, a_{n2}, \dots) \in (l^v(p,t)^+$ for each n and so, by Lemma 5(i) and lemma 6, $A_n \in (l^v(p,t)^*,\ \forall n$. Therefore, by Lemma 1 there exists $M > 1$ and $G > 1$ such that $|\, A_n(x)\,|^{\rho_n} \leq G,\ \forall n$ and $x \in l^v(p,t)$ with $g(x) \leq 1/M$.

Thus,

$$|\sum_k G^{-1/q_n}(a_{nk} v_k^{-1})\, x_k| \leq 1,\ (n = 1, 2, \dots) \text{ if } g(x) \leq 1/M.$$

Now, write $\Lambda = (G^{-\frac{1}{q_n}}(a_{nk} v_k^{-1}))$ and choose any $x \in l^v(p,t)$. By the continuity of scalar multiplication on $l^v(p,t)$, there is a $K \geq 1$, such that $g(K^{-1}x) \leq 1/M$, whence

$$|\sum_k G^{-1/q_n}(a_{nk} v_k^{-1})\, x_k| \leq K,\ \forall n.$$

Thus, we see that $\Lambda \in (l^v(p,t) \colon l_\infty)$ and so there exists $D > 1$ such that

$$\sup_n \sum_k k^{t(s_k-1)} D^{-s_k} |G^{-1/q_n}(a_{nk} v_k^{-1})|^{s_k} < \infty.$$

Writing $B = GD$ and using the fact that $D^{q_n} \leq D,\ \forall n$ we obtain (6).

Theorem B: (i) Suppose $0 < p_k \leq 1,\ p_k^{-1} + s_k^{-1} = 1$ for every $k \in N$ and q be bounded. Then $A \in (l^v(p,t) \colon c_0(q))$ if and only if

$$|a_{nk} v_k^{-1}|^{q_n} \to 0 \quad (n \to \infty), \text{ for each } k \in N, \tag{7}$$

$$\lim_M \sup_n \{\sup_k k^{t/p_k} |\, a_{nk} v^{-1}| M^{-1/p_k}\}^{q_n} \to 0 \tag{8}$$

(ii) Let $1 < p_k \leq \infty,\ p_k^{-1} + s_k^{-1} = 1$ for every $k \in N$ and q be bounded. Then $A \in (l^v(p,t) \colon c_0(q))$ if and only if (7) holds and for each $D \geq 1$,

$$\lim_B \limsup_n \{\sum_k k^{t(s_k-1)} D^{-s_k/q_n} B^{-s_k} |a_{nk} v_k^{-1}|^{s_k}\}^{q_n} = 0 \tag{9}$$

Proof: (i) Let $A \in (l^v(p,t): c_0(q))$. Since $(l^v(p,t): c_0(q) \subset (l^v(p,t): l_\infty(q)$ then as in the preceding theorem we must have $A_n \in (l^v(p,t)^*$ and

$$||A_n||_M = \sup_k M^{-1/p_k} |a_{nk} v_k^{-1}| k^{t/p_k}.$$

whenever $||A_n||_M$ is defined, for each $n \in N$. Then by Lemma 2 part 3, (5) must hold. (4) is easily obtained since $x = e^{(k)} \in l^v(p,t)$ for each $k = 1, 2, 3, \dots$.

Conversely, if (7) and (8) hold we can show that $A_n \in (l^v(p,t)^*$, with

$$||A_n||_M = \sup_k M^{-1/p_k} |a_{nk} v_k^{-1}| k^{t/p_k}.$$

whenever $||A_n||_M$ is defined, for each $n \in N$; also $(e^{(k)})$ is a basis in $l^v(p,t)$. Then Theorem A(i) implies that $A \in (l^v(p,t): c_0(q))$.

(ii) Define A_n by

$$A_n(x) = \sum_k a_{nk} x_k$$

on $l^v(p,t)$ for each $n \in N$; and consider the proof of necessity. Thus, let $A \in (l^v(p,t): c_0(q))$. Obviously we have (4) as in Theorem A(ii) and we see that $A_n \in (l^v(p,t)^*$ $\forall n$. If $A \in (l^v(p,t): c_0(q))$ then

$$D^{1/q_n}(a_{nk} v_k^{-1}) \in \left(l^v(p,t): c_0(q)\right), \ \forall D > 1.$$

So, it is enough to show that (9) holds for $D = 1$. Since $c_0(q) \subset l_\infty(q)$ and using Lemma 3 (i), there exists $B > 1$ such that

$$T_n = \sum_k k^{t(s_k - 1)} B^{-\frac{H}{s_k}} |a_{nk}|^{s_k} \le 1, \text{ for every } n \in N.$$

Choose any n, and define

$$x_k^{(n)} = B^{-\frac{H}{s_k}} sgn(a_{nk} v_k^{-1}) |\, a_{nk} v_k^{-1}|^{s_k - 1} \cdot k^{t(s_k - 1)}, \text{for each } k;$$

Then,

$$g^H\left(x^{(n)}\right) = \sum_k k^{-t} k^{t(s_k-1)^{p_k}} B^{(-Hs_k)p_k} |\, a_{nk} v_k^{-1}|^{s_k/p_k}$$

$$= \sum_k k^{t(s_k-1)^{p_k}} B^{(-Hs_k)p_k} |\, a_{nk} v_k^{-1}|^{s_k}$$

$$\leq B^{-H} \sum_k k^{t(s_k-1)^{p_k}} B^{(-Hs_k)} |\, a_{nk} v_k^{-1}|^{s_k}$$

$$\leq B^{-H}.$$

And,

$$A_n(x^n) = \sum_k \, a_{nk} v_k^{-1} \, x_k^{(n)}$$

$$= \sum_k \, a_{nk} \, k^{t(s_k-1)} B^{(-Hs_k)} |\, a_{nk} v_k^{-1}|^{s_k-1} \mathrm{sgn}(\, a_{nk} v_k^{-1})$$

$$= T_n$$

whence $||A_n||_B \geq T_n$ for each n. By Lemma 2b (1) we must have $\lim_B \limsup_n (||A_n||_B)^{q_n} = 0$, whence (y) holds with $D = 1$.

Sufficiency: Let (11) and (9) hold $\forall\, D \geq 1$. It follows that $A_n \in (l^v(p,t)^* \;\forall n \in N$. Since $(e^{(k)})$ is a basis in $l^v(p,t)$ and using Lemma 2 b(1) it is enough to show that $\lim_B \limsup_n (||A_n||_B)^{q_n} = 0$.

Now choose $\varepsilon > 0$ such that $0 < \varepsilon \leq 1$ and $D > 2/\varepsilon$. There exists $B > 1$ and m such that

$$(\sum_k k^{t(s_k-1)} D^{-s_k/q_n} B^{-s_k} |\, a_{nk} v_k^{-1}|^{s_k})^{q_n} < \frac{\varepsilon}{2} \text{ if } n \geq m.$$

Then if $g(x) \leq 1/B$ and if $n \geq m$ we have

$$|A_x(x)|^{q_n} \leq (\sum_k |\, a_{nk} v_k^{-1}|\, D^{1/q_n}\, B^{-1} B \;\, D^{-1/q_n} k^{t/p_k} \;\, k^{-t/p_k} |x_k|)^{q_n}$$

$$\leq (\sum_k |\, a_{nk} v_k^{-1}|^{s_k}\, D^{s_k/q_n}\, B^{-s_k}\, k^{t(s_k-1)} + D^{-p_k/q_n} \;\, k^{-t} |x_k|^{p_k})^{q_n}$$

$$\leq (\sum_k |\, a_{nk} v_k^{-1}|^{s_k}\, D^{\frac{s_k}{q_n}}\, B^{-s_k}\, k^{t(s_k-1)} \;\;)^{q_n} + (D^{-\frac{1}{q_n}} B^H g^H(x))^{q_n}$$

$$\leq \varepsilon/2 \; + (D^{-\frac{1}{q_n}} B^H g^H(x)^{q_n}$$

$$< \varepsilon$$

which completes the proof.

Conclusion: These characterizations are generalizations of Bilgin (see [8]) and fill the gap in the existing literature. The results obtained here also throw light on the ways for further generalizations.

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