# INEXTENSIBLE FLOWS OF STRICTION CURVE OF RULED SURFACE

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## ABSTRACT

*In this paper, we study inextensible flows of striction curve of ruled surface. Necessary and sufficient conditions for an inelastic striction curve flow are expressed as a partial differential equation involving 1st curvature and 2nd curvature.*

**KEYWORDS**: Inextensible flows; Ruled surface .

## 1. INTRODUCTION

Recently, the study of the motion of inelastic curves have an important role. The time evolution of a curve represented by its corresponding flow. The flow of a curve is said to be inextensible if, firstly its arc length is preserved and secondly its intrinsic curvature is preserved. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve. Some movement in nature is inspired to examine flow of curves as snake and elephant's trunk movement. For example, both Chirikjian and Burdick [1] and Mochiyama et al. [2] study the shape control hyper-redundant, or snake-like robots. Inextensible curve and surface flows emerge many problems in computer vision [3] and computer animation [4].

Particularly, inextensible time evolution of curves and surfaces is examined mathematically. Significant methods of this article developed by Gage and Hamilton [5], and Grayson [6] for studying the shrinking of closed plane curves to a circle via the heat equation. In [7] Gage also studies area preserving evolution of inelastic plane curves. In [8,9] Know et al. study evolution of inelastic plane curves, and inextensible flows of curves and developable surfaces.

In this paper, we study inextensible flows of striction curves of ruled surfaces according to Blaschke frame. Necessary and sufficient conditions for an inelastic striction curve of ruled surface flow are expressed as a partial differential equation involving the 1st curvature and 2nd curvature.

## 2. DIFFERENTIAL GEOMETRY OF RULED SURFACE IN $E^3$

Let $I$ be an open interval in the real line $IR$. Let $k = k(s)$ be a curve in $E^3$ defined on $I$ and $q = q(s)$ be a unit direction vector of an oriented line in $IR^3$. Then we have the following parametrization for a ruled surface

$$\varphi_q(s,v) = \vec{k}(s) + v\,\vec{q}(s) \tag{2.1}$$

The parametric $s$-curve of this surface is a straight line of the surface which is called ruling. For $v = 0$, the parametric $v$-curve of this surface is $\vec{k} = \vec{k}(s)$ which is called base curve or generating curve of the surface. In particular, if $\vec{q}$ is constant, the ruled surface is said to be cylindrical, and non-cylindrical otherwise[10].

The striction point on a ruled surface is the foot of the common normal between two consecutive rulings. The set of the striction points constitute a curve $\vec{c} = \vec{c}(s)$ lying on the ruled surface and is called striction curve. The parametrization of the striction curve $\vec{c} = \vec{c}(s)$ on a ruled surface is given by

$$\vec{c}(s) = \vec{k}(s) - \frac{\left\langle d\vec{q},\, d\vec{k} \right\rangle}{\left\langle d\vec{q},\, d\vec{q} \right\rangle}\vec{q}\,.$$

(2.2)

So that, the base curve of the ruled surface is its striction curve if and only if $\left\langle d\vec{q},\, d\vec{k} \right\rangle = 0$.

The distribution parameter (or drall) of the ruled surface in (2.1) is given as

$$\delta_q = \frac{\left\langle d\vec{k}, \vec{q} \times d\vec{q} \right\rangle}{\left\langle d\vec{q}, d\vec{q} \right\rangle}$$

(2.3)

If $\delta_q = 0$, then the normal vectors of the ruled surface are collinear at all points of the same ruling and at the nonsingular points of the ruled surface the tangent planes are identical. We then say that the tangent plane contacts the surface along a ruling. Such a ruling is called a torsal ruling. If $\delta_q \neq 0$, then the tangent planes are distinct at all points of the same ruling which is called nontorsal.

A ruled surface whose all rulings are torsal is called a developable ruled surface. The remaining ruled surfaces are called skew ruled surfaces. Thus, from (2.3) a ruled surface is developable if and only if at all its points the distribution parameter $\delta_q = 0$ [10,11].

Let $\left\{\vec{q}, \vec{h} = \frac{d\vec{q}/ds}{\|d\vec{q}/ds\|}, \vec{a} = \vec{q} \times \vec{h}\right\}$ be a moving othonormal trihedron making a spatial motion along a closed space curve $\vec{k}(s)$, $s \in \mathbb{R}$, in $E^3$. In this motion, an oriented line fixed in the moving system generates a closed ruled surface called closed trajectory ruled surface (CTRS) in $E^3$ [12]. A parametric equation of a closed trajectory ruled surface generated by $\vec{q}$-axis is

$$\begin{aligned} &\varphi_q(s,v) = \vec{k}(s) + v\,\vec{q}(s), \\ &\varphi(s+2\pi, v) = \varphi(s,v), \quad s, v \in \mathbb{R} \end{aligned} \tag{2.4}$$

Consider the moving orthonormal system $\{\vec{q}, \vec{h}, \vec{a}\}$. Then, the axes of the trihedron intersect at the striction point of $\vec{q}$-generator of $\varphi_q$-CTRS. The structral equations of this motion are

$$\begin{cases} d\vec{q} = \kappa_1 \vec{h} \\ d\vec{h} = -\kappa_1 \vec{q} + \kappa_2 \vec{a} \\ d\vec{a} = -\kappa_2 \vec{h} \end{cases}$$

(2.5)

and

$$\frac{dc}{ds} = \cos\sigma \vec{q} + \sin\sigma \vec{a}$$

(2.6)

where $c = c(s)$ is the striction line of $\varphi_q$-CTRS and the differential forms $\kappa_1$, $\kappa_2$ and $\sigma$ are the natural curvature, the natural torsion and the striction of $\varphi_q$-CTRS, respectively[10,12]. Here, the striction is restricted as $-\pi/2 < \sigma < \pi/2$ fort he orientation on $\varphi_q$-CTRS and $s$ is the length of the striction line.

The pole vector and the Steiner vector of the motion are given by

$$\vec{p} = \frac{\vec{\psi}}{\|\vec{\psi}\|}, \quad \vec{d} = \oint \vec{\psi}$$

(2.7)

respectively, where $\vec{\psi} = \kappa_2 \vec{q} + \kappa_1 \vec{a}$ is the instantaneous Pfaffian vector of the motion.

## 3.INEXTENSIBLE FLOWS OF STRICTION CURVE OF RULED SURFACE

Throughout this study, we assume that $c(s):[0,l]\times[0,t_\infty] \to \mathbb{R}^3$ is a one parameter family of smooth curves in space $\mathbb{R}^3$. Let $u$ be the curve parametrization variable, $u \in [0,l]$.

The arclenght of striction curve $c$ is given by

$$s = \int_0^u \left|\frac{\partial c}{\partial u}\right| du = \int_0^u v\, du$$

(3.1)

where

$$\left|\frac{\partial c}{\partial u}\right| = \left|\left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\right\rangle\right|^{1/2}.$$

(3.2)

The operator $\frac{\partial}{\partial s}$ is given in terms of $u$ by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$$

where

$$v = \left|\frac{\partial c}{\partial u}\right|.$$

The arclenght parameter is $ds = v\, du$.

Any flow of $c$ can be represented as

$$\frac{\partial c}{\partial t} = f_1\vec{q} + f_2\vec{h} + f_3\vec{a}$$

(3.3)

The arclength is given up to a constant by

$$s(u,t) = \int_0^u v\,du$$

In the real space the requirement that the curve not be exposed to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0$$

(3.4)

for $u \in [0,l]$.

**Definition 3.1.** *A striction curve of ruled surface evolution* $c(u,t)$ *and* $\frac{\partial c}{\partial t}$ *flow of this curve in* $\mathbb{R}^3$ *are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial c}{\partial u}\right| = 0 .$$

**Lemma 3.2.** *Let* $\frac{\partial c}{\partial u}$ *be a smooth flow of the striction curve* $c$*. The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v \kappa_1$$

(3.5)

**Proof.** Suppose that $\frac{\partial c}{\partial u}$ be a smooth flow of the striction curve $c$. Considering definition of $c$, we have

$$v^2 = \left\langle \frac{\partial c}{\partial u}, \frac{\partial c}{\partial u} \right\rangle$$

(3.6)

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since and are independent coordinates. Then, by differentiating of formula (3.1) we get

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial c}{\partial u}, \frac{\partial c}{\partial u} \right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ we

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial c}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial c}{\partial t}\right) \right\rangle$$

From equation (3.3) , we obtain

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial c}{\partial u}, \frac{\partial}{\partial u}(f_1\vec{q} + f_2\vec{h} + f_3\vec{a}) \right\rangle.$$

By the formula of Blaschke derivative , we have

$$\frac{\partial v}{\partial t} = \left\langle \vec{q}, \left(\frac{\partial f_1}{\partial u} - f_2 v\kappa_1\right)\vec{q} + \left(f_1 v\kappa_1 + \frac{\partial f_2}{\partial u} - f_3 v\kappa_2\right)\vec{h} + \left(f_2 v\kappa_2 + \frac{\partial f_3}{\partial u}\right)\vec{a} \right\rangle.$$

Thus we have

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v\kappa_1.$$

**Theorem 3.3.** *Let* $\frac{\partial c}{\partial u} = f_1\vec{q} + f_2\vec{h} + f_3\vec{a}$ *be a smooth flow of the striction curve* $c$ *of ruled surface. The flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = f_2\kappa_1.$$

(3.7)

**Proof.** Let $\frac{\partial c}{\partial u}$ be extensible. From Eq. (3.4), we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}du=\int_0^u (\frac{\partial f_1}{\partial u}-f_2 v\kappa_1)du=0$$

(3.8)

$\forall u\in[0,l]$. $\frac{\partial f_1}{\partial u}=f_2 v\kappa_1$, or $1/v\frac{\partial f_1}{\partial u}=f_2\kappa_1$, or $\frac{\partial f_1}{\partial s}=f_2\kappa_1$ as claimed. We suppose that $v=1$ and the local coordinate $u$ corresponds to the curve arc length $s$. Now we give following lemma that necessary.

**Lemma 3.4.**

$$\frac{\partial\vec{q}}{\partial t}=\left(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2\right)\vec{h}+\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{a},$$

(3.9)

$$\frac{\partial\vec{h}}{\partial t}=-(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2)\vec{q}+\psi\vec{a},$$

(3.10)

$$\frac{\partial\vec{a}}{\partial t}=-\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{q}-\psi\vec{h},$$

(3.11)

*where* $\psi=\left\langle\frac{\partial\vec{h}}{\partial t},\vec{a}\right\rangle.$

**Proof.** Considering definition of $c$, we have

$$\frac{\partial\vec{q}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial c}{\partial s}=\frac{\partial}{\partial s}(f_1\vec{q}+f_2\vec{h}+f_3\vec{a}).$$

Using the Blaschke equations, we get

$$\frac{\partial\vec{q}}{\partial t}=\left(\frac{\partial f_1}{\partial s}-f_2\kappa_1\right)\vec{q}+\left(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2\right)\vec{h}+\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{a}.$$

(3.12)

Substituting (3.7) in (3.12), we have

$$\frac{\partial\vec{q}}{\partial t}=\left(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2\right)\vec{h}+\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{a}.$$

Now differentiate the Blaschke frame by t:

$$0=\frac{\partial}{\partial t}\langle\vec{q},\vec{h}\rangle=\left\langle\frac{\partial\vec{q}}{\partial t},\vec{h}\right\rangle+\left\langle\vec{q},\frac{\partial\vec{h}}{\partial t}\right\rangle= f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2+\left\langle\vec{q},\frac{\partial\vec{h}}{\partial t}\right\rangle,$$
$$0=\frac{\partial}{\partial \overline{t}}\langle\vec{q},\vec{a}\rangle=\left\langle\frac{\partial\vec{q}}{\partial t},\vec{a}\right\rangle+\left\langle\vec{q},\frac{\partial\vec{a}}{\partial t}\right\rangle= f_2\kappa_2+\frac{\partial f_3}{\partial s}+\left\langle\vec{q},\frac{\partial\vec{a}}{\partial t}\right\rangle,$$
$$0=\frac{\partial}{\partial \overline{t}}\langle\vec{h},\vec{a}\rangle=\left\langle\frac{\partial\vec{h}}{\partial t},\vec{a}\right\rangle+\left\langle\vec{h},\frac{\partial\vec{a}}{\partial \overline{t}}\right\rangle=\psi+\left\langle\vec{h},\frac{\partial\vec{a}}{\partial \overline{t}}\right\rangle.$$

Considering $\left\langle\frac{\partial\vec{h}}{\partial t},\vec{h}\right\rangle=\left\langle\frac{\partial\vec{a}}{\partial t},\vec{a}\right\rangle=0$ and from above statement, we obtain

$$\frac{\partial\vec{h}}{\partial t}=-(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2)\vec{q}+\psi\vec{a},$$
$$\frac{\partial\vec{a}}{\partial \overline{t}}=-\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{q}-\psi\vec{h},$$

where $\psi=\left\langle\frac{\partial\vec{h}}{\partial t},\vec{a}\right\rangle$.

The following theorem states the conditions on the 1st curvature and 2nd curvature for the striction curve flow $c(s,t)$ to be inextensible.

**Theorem 3.5.** *Suppose* $\frac{\partial c}{\partial t}=f_1\vec{q}+f_2\vec{h}+f_3\vec{a}$ *is inextensible. Then, the following system of partial differential equations holds:*

$$\frac{\partial\kappa_1}{\partial t}=\frac{\partial}{\partial s}(f_1\kappa_1)+\frac{\partial^2 f_2}{\partial s^2}+\frac{\partial}{\partial s}(f_3\kappa_2)-f_2\kappa_2^2+\frac{\partial f_3}{\partial s}\kappa_2,$$
$$\kappa_1\psi=\frac{\partial(f_2\kappa_2)}{\partial s}+\frac{\partial^2 f_3}{\partial s^2}+\kappa_2(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2),$$
$$\frac{\partial\kappa_2}{\partial t}=(f_2\kappa_2+\frac{\partial f_3}{\partial s})\kappa_1+\frac{\partial\psi}{\partial s}.$$

(3.13)

**Proof.** Noting that $\frac{\partial}{\partial s}\frac{\partial\vec{q}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial\vec{q}}{\partial s}$,

$$\frac{\partial}{\partial s}\frac{\partial \vec{q}}{\partial t}=\frac{\partial}{\partial s}\left[\left(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f\kappa_2\right)\vec{h}+\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{a}\right]$$

$$=\left(\frac{\partial(f_1\kappa_1)}{\partial s}+\frac{\partial^2 f_2}{\partial s^2}+\frac{\partial(f_3\kappa_2)}{\partial s}\right)\vec{h}-\left(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2\right)(-\kappa_1\vec{q}+\kappa_2\vec{a})$$

$$+\left(\frac{\partial(f_2\kappa_2)}{\partial s}+\frac{\partial^2 f_3}{\partial s^2}\right)\vec{a}+\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)(-\kappa_2\vec{h})$$

while

$$\frac{\partial\kappa_1}{\partial t}=\frac{\partial}{\partial s}(f_1\kappa_1)+\frac{\partial^2 f_2}{\partial s^2}+\frac{\partial}{\partial s}(f_3\kappa_2)-f_2\kappa_2^2+\frac{\partial f_3}{\partial s}\kappa_2$$

and

$$\kappa_1\psi=\frac{\partial(f_2\kappa_2)}{\partial s}+\frac{\partial^2 f_3}{\partial s^2}+\kappa_2(f_1\kappa_1+\frac{\partial f_2}{\partial s}-f_3\kappa_2).$$

Since $\frac{\partial}{\partial s}\cdot\frac{\partial\vec{a}}{\partial t}=\frac{\partial}{\partial t}\cdot\frac{\partial\vec{a}}{\partial s}$, we get

$$\frac{\partial}{\partial s}\cdot\frac{\partial\vec{a}}{\partial t}=\frac{\partial}{\partial s}\left(-\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\vec{q}-\psi\vec{h}\right)$$

$$=\left(\frac{\partial}{\partial s}(-f_2\kappa_2)+\frac{\partial^2 f_3}{\partial s^2}\right)\vec{q}-\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\kappa_1 h-\frac{\partial\psi}{\partial s}\vec{h}-\psi(-\kappa_1\vec{q}+\kappa_2\vec{a})$$

while

$$\frac{\partial}{\partial t}\cdot\frac{\partial\vec{a}}{\partial s}=\frac{\partial}{\partial t}(-\kappa_2\vec{h})=-\frac{\partial\kappa_2}{\partial t}\vec{h}-\kappa_2(-\kappa_1\vec{q}+\kappa_2\vec{a}).$$

Therefore

$$\frac{\partial\kappa_2}{\partial t}=\left(f_2\kappa_2+\frac{\partial f_3}{\partial s}\right)\kappa_1+\frac{\partial\psi}{\partial s}.$$

## 4. CONCLUSIONS

Inextensible time evolution of curves and surfaces have an important role in computer vision, robotics and physical science. In this paper inextensible flows of striction curve of ruled surface according to Blaschke frame have given by considering important role of Euclidean geometry.

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