ON THE NON-HOMOGENEOUS CUBIC EQUATION WITH FIVE UNKNOWNS $x^2+xy-y^2-z-w=T^3$.

S.Vidhyalakshmi [1], M. A. Gopalan [2] and K. Lakshmi [3]
[1,2,3] Department of Mathematics,Shrimati Indira Gandhi College,Trichy, India.

ABSTRACT

We obtain infinitely many non-zero integer solutions (x, y, z, w, T) *satisfying the non-homogeneous cubic equation with five unknowns given by* $x^2+xy-y^2-z-w=T^3$. *Various interesting relations between the solutions and special numbers are presented*

KEYWORDS:

Non-homogeneous cubic equation, Integral solutions, Polygonal numbers, Pyramidal numbers, Centered pyramidal numbers, Four dimensional pentagonal number.

MSC 2000 Mathematics subject classification: 11D25.

NOTATIONS:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

OH_n - Octahedral number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{4,n,5}$ -Four dimensional pentagonal number of rank n

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3]. For illustration, one may refer [4-11] for homogeneous and non-homogeneous cubic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous cubic equation with five unknowns given by $x^2+xy-y^2-z-w=T^3$

A few relations between the solutions and the special numbers are presented.

Initially, the following two sets of solutions in (x,y,z,w,T) satisfy the given equation:

$(4k^2, 2k, 2(4k^4-k^2)+p, 2(4k^4-k^2)-p, 2k)$,

$(-2k\alpha^2, -2k, -2k^2(1-\alpha^4)+p, -2k^2(1-\alpha^4)-p, -2\alpha k)$

However we have other patterns of solutions, which are illustrated below:

Method of Analysis:

The Diophantine equation representing the non- homogeneous cubic equation is given by

$$x^2+xy-y^2-z-w=T^3 \quad (1)$$

Introduction of the transformations

$$x=u+v, y=u-v, z=2uv+p, w=2uv-p \quad (2)$$

in (1) leads to

$$u^2-v^2=T^3 \quad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

Approach1:

The solution to (3) is obtained as

$$u=a(a^2-b^2), v=b(a^2-b^2), T=a^2-b^2 \quad (4)$$

In view of (2) and (4), the corresponding values of (x,y,z,w,T) are represented by

$$\left.\begin{aligned} x&=(a+b)(a^2-b^2)\\ y&=(a-b)(a^2-b^2)\\ z&=2ab(a^2-b^2)^2+p\\ w&=2ab(a^2-b^2)^2-p\\ T&=(a^2-b^2) \end{aligned}\right\} \quad (5)$$

The above values of x, y, z, w **and** T **satisfies the following relations:**

1. $x(a+2,a+1)+y(a+2,a+1)+T(a+2,a+1)-4T_{4,a}-32CP_{a,6}+48(OH_a)\equiv 0(\mathrm{mod}\,15)$

2. The following expressions are nasty numbers:

(a) $3p[2z(a,b)-x^2(a,b)-y^2(a,b)]$.

(b) $x(2a,a)+y(2a,a)-6SO_a+6PR_a$.

3. The following expressions are cubic integers

(a) $9[x(2a,a)+y(2a,a)+z(2a,a)+w(2a,a)+T(2a,a)-6P_a^5]$.

(b) $9[4x(2a,a)+4y(2a,a)+z(2a,a)+w(2a,a)-36(SO_{a^3}-T_{4,a})]$

4. $16[x(a,1)-y(a,1)+w(a,1)+4CP_{a,6}-4T_{3,a}+1]$ is a quintic integer

5. $x(a+1,a)+y(a+1,a)+T(a+1,a)-8T_{3,a}-8CP_{a,6}+12(OH_a)\equiv 0(mod\,3)$

6. $x(2^{4n},2^{2n})+y(2^{4n},2^{2n})=2(j_{12n}-j_{8n})$

7. $9[4x(2a,a)+4y(2a,a)+z(2a,a)+w(2a,a)-36(SO_{a^3}-T_{4,a})]$

8. $T(2^{2n+1},2^{2n})=3KY_{2n}-3J_{2n+1}$

9. $z(2a,a)-w(2a,a)-x(2a,a)-y(2a,a)+12CP_{a,6}\equiv 0(\mathrm{mod}\,2)$

10. $y(a,1)-x(a,1)+T(a,1)+T_{4,a}=1$

Approach2:

The assumption

$$u=UT, v=VT \tag{6}$$

in (3) yields to

$$U^2-V^2=T \tag{7}$$

Taking $T=-t^2$ (8)

in (7), we get

$$U^2+t^2=V^2 \tag{9}$$

(i) Then the solution to (9) is given by

$$t=2\alpha\beta, V=\alpha^2+\beta^2, U=\alpha^2-\beta^2,\ \alpha>\beta>0 \quad \text{(OR)} \tag{10}$$

$$U=2\alpha\beta, V=\alpha^2+\beta^2, t=\alpha^2-\beta^2,\ \alpha>\beta>0 \tag{11}$$

From (6), (8) and (10) we get

$$\left.\begin{aligned} u&=-4\alpha^2\beta^2(\alpha^2-\beta^2)\\ v&=-4\alpha^2\beta^2(\alpha^2+\beta^2)\\ T&=-4\alpha^2\beta^2 \end{aligned}\right\} \tag{12}$$

In view of (12) and (2), we get the corresponding integral solution of (1).as

$$x = -8\alpha^4\beta^2$$
$$y = -8\alpha^2\beta^4$$
$$z = 32\alpha^4\beta^4(\alpha^4 - \beta^4) + p$$
$$w = 32\alpha^4\beta^4(\alpha^4 - \beta^4) - p$$
$$T = -4\alpha^2\beta^2$$

Remark: 1

By considering (6), (8), (11) and (2), we get the corresponding integral solution to (1).as

$$x = -(\alpha^2 - \beta^2)^2(\alpha + \beta)^2$$
$$y = (\alpha^2 - \beta^2)^2(\alpha - \beta)^2$$
$$z = 4\alpha\beta(\alpha^2 + \beta^2)(\alpha^2 - \beta^2)^4 + p$$
$$w = 4\alpha\beta(\alpha^2 + \beta^2)(\alpha^2 - \beta^2)^4 - p$$
$$T = -(\alpha^2 - \beta^2)^2$$

Properties:

1. $x(2a,a) + y(2a,a) + z(2a,a) - w(2a,a) + 72CP_{a^3,6} \equiv 0(mod\,2)$
2. $x(2a,a) - y(2a,a) + 360(P_a^5)^2 = 90T_{4,a}(2P_a^8 + 2T_{3,a} - T_{4,a})$
3. $3[72(2T_{3,a} - T_{4,a}) - 36PR_a - T(a-1, a+1)]$ is a cubic integer
4. $z(2a,a) - w(2a,a) - 2p + 42F_{4,a,5} - 21CP_{a,6} - 14T_{4,a}$ is a biquadratic integer
5. $z(a,1) + w(a,1) - 16CP_{a,6}(T_{4,a} - 1)^4 = 0$

(ii) Now, rewrite (9) as,

$$U^2 + t^2 = 1 * V^2 \quad (13)$$

Also 1 can be written as

$$1 = (-i)^n (i)^n \quad (14)$$

Let $V = a^2 + b^2$ (15)

Substituting (14) and (15) in (13) and using the method of factorisation, define,

$$(U + it) = i^n (a + ib)^2 \quad (16)$$

Equating real and imaginary parts in (16) we get

$$\left.\begin{aligned} U &= \cos\frac{n\pi}{2}(a^2 - b^2) - 2ab\sin\frac{n\pi}{2} \\ t &= 2ab\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}(a^2 - b^2) \end{aligned}\right\} \quad (17)$$

In view of (2), (6), (8) and (17), the corresponding values of x, y, z, w, T are represented as

$$x = T[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2} + (a^2+b^2)]$$

$$y = T[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2} - (a^2+b^2)]$$

$$z = 2T^4(a^2+b^2)[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2}] + p$$

$$w = 2T^4(a^2+b^2)[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2}] - p$$

$$T = -[\sin\frac{n\pi}{2}(a^2-b^2) + 2ab\cos\frac{n\pi}{2}]^2$$

(iii) 1 can also be written as

$$1 = \frac{((m^2-n^2)+i2mn)((m^2-n^2)-i2mn)}{(m^2+n^2)^2} \tag{18}$$

Substituting (15) and (18) in (13) and using the method of factorisation, define,

$$(U+it) = \frac{(m^2-n^2)+i2mn)}{(m^2+n^2)^2}(a+ib)^2 \tag{19}$$

Equating real and imaginary parts in (19) we get

$$\left.\begin{aligned} U &= \frac{1}{m^2+n^2}\{(m^2-n^2)(a^2-b^2) - 4mnab\} \\ t &= \frac{1}{m^2+n^2}\{2ab(m^2-n^2) + 2mn(a^2-b^2)\} \end{aligned}\right\} \tag{20}$$

In view of (2), (6), (8) and (20), the corresponding values of x, y, z, w, T are represented as

$$x = (m^2+n^2)T[(m^2-n^2)(A^2-B^2) - 4mnAB + (m^2+n^2)(A^2+B^2)]$$

$$y = (m^2+n^2)T[(m^2-n^2)(A^2-B^2) - 4mnAB - (m^2+n^2)(A^2+B^2)]$$

$$z = 2T^4(m^2+n^2)[(A^4-B^4)(m^4-n^4) - 4mnAB] + p$$

$$w = 2T^4(m^2+n^2)[(A^4-B^4)(m^4-n^4) - 4mnAB] - p$$

$$T = -(m^2+n^2)^2[2AB(m^2-n^2) + 2mn(A^2-B^2)]^2$$

Remark2: By writing 1 as

$$1 = \frac{(2mn+i(m^2-n^2)(2mn-i(m^2-n^2)}{(m^2+n^2)^2}$$

and performing the same procedure as above the corresponding integral solution to (1) can be obtained

Approach3:

Equation (3) can be written as

$$(u-v)(u+v) = 1\times T^3 \tag{21}$$

Writing (21) as a set of double equations in two different ways as shown below, we get

Set1: $u+v = T^3,\ u-v = 1$

Set2: $u-v=T^3,\ u+v=1$

Solving **set1,** the corresponding values of u, v *and* T are given by

$$u=4k^3+6k^2+3k+1, v=4k^3+6k^2+3k, T=2k+1 \tag{22}$$

In view of (22) and (2), the corresponding solutions to (1) obtained from set1 are represented as shown below:

$$x=8k^3+12k^2+6k+1$$
$$y=1$$
$$z=2(4k^3+6k^2+3k+1)(4k^3+6k^2+3k)+p$$
$$w=2(4k^3+6k^2+3k+1)(4k^3+6k^2+3k)-p$$
$$T=2k+1$$

Properties:

1. $x(a)+y(a)-24P_a^4 \equiv 0 \pmod 2$
2. $6[2x(a)+2y(a)+z(a)+w(a)+T(a)-4T_{3,a}+T_{4,a}-1]$ is a nasty integer.
3. $4[x(a)+y(a)-24P_a^4-5CP_{a,6}+3CP_{a,10}]$ is a cubic integer.
4. $8j_{6n}+12j_{4n}+6j_{2n}+3J_{2n+1}-x(2^{2n},2^{2n})+y(2^{4n},2^{2n})-T(2^{4n},2^{2n}) \equiv 0 \pmod{26}$
5. $z(a)-w(a)+y(a)-2p=1$

Remark3:

Similarly, the solutions corresponding to set2 can also be obtained.

Approach4:

Substituting, $T=a^2-b^2$

in (3) and writing it as a system of double equations as

$$u+v=(a+b)^3$$
$$u-v=(a-b)^3$$

and solving we get

$$\left.\begin{aligned} u&=a^3+3ab^2 \\ v&=3a^2b+b^3 \end{aligned}\right\} \tag{26}$$

In view of (26) and (2), the corresponding solutions to (1) are represented as shown below:

$$x=(a+b)^3$$
$$y=(a-b)^3$$
$$z=2(a^3+3ab^2)(3a^2b+b^3)+p$$
$$w=2(a^3+3ab^2)(3a^2b+b^3)-p$$
$$T=a^2-b^2$$

Properties:

1. $3[x(a,a)+y(a,a)+z(a,a)-w(a,a)-8CP_{a,6}]$ is a nasty number
2. $x(a,1)+y(a,1)-4CP_{a,3}-6T_{4,a}+2T_{8,a}=0$
3. $x(a,a)+T(a,a)+z(a,a)-w(a,a)-2$ is a cubical integer
4. $z(a,a)+w(a,a)+x(a,a)+y(a,a)-144P_a^5+72T_{4,a}=0$
5. $x(a,1)+y(a,1)-6P_a^4-4T_{4,a}+2T_{9,a}=0$

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Conclusion:

In conclusion, one may search for different patterns of solutions to (1) and their corresponding properties.

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