SPECTRAL CHARACTERIZATIONS OF ELEMENTS IN SEMISIMPLE BANACH ALGEBRAS

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ABSTRACT

We recall some definitions and properties of spectrum and spectral radius in Banach algebras and introduce some spectral characterizations of elements in semisimple Banach algebras. We investigate under what conditions, in a semisimple Banach algebra A the number of elements in the spectrum of ax is less than or equal to the number of elements in the spectrum of x for all x in an arbitrary neighbourhood of the identity. We also investigate the conditions that an element x is of rank one and idempotent.

KEYWORDS: Banach Algebras; Spectral Radius; Spectrum; Semisimple Algebra
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1. INTRODUCTION

In [7] J. Puhl defined rank one element of a semiprime Banach algebra. It is also proved that the trace of a nuclear element is zero. In paper [4], T. Mouton and H. Raubenheimer gave a spectral characterization of rank one elements and socle of a semisimple Banach algebra. In [2] it is shown that the trace and the determinant on a semisimple Banach algebra A can be defined in a purely spectral and analytic way. In fact these two notions are well defined on the socle of A, and which is by definition the sum of all minimal left ideals (or minimal right ideals) of A. It is well known that the socle is a two sided algebraic ideal, then in particular all its elements have finite spectrum.

2. PRELIMINARY

Throughout A (or B) will be a complex Banach algebra with unit 1 over the field of complex numbers $\mathbb{C}$. A set $\mathrm{J} \subset \mathrm{A}$ is called a left (right) ideal in A if J is a subspace of A and $ax \in \mathrm{J}$ ($xa \in \mathrm{J}$) for all $x \in \mathrm{J}, a \in \mathrm{A}$. J is a two-sided ideal in A if J is both left and right ideal in A. An ideal $\mathrm{J} \subset \mathrm{A}$ (left, right or two-sided) is called proper if $\mathrm{J} \neq \mathrm{A}$. J is a maximal ideal if J is proper and if the only proper ideal containing J is J itself. The intersection of all maximal ideals in A is called (Jacobson) radical of A and is denoted by Rad A. An algebra A is called semisimple if Rad $A = \{0\}$. An algebra A be semiprime if $xAx = \{0\}$ implies $x = 0$ holds for all $x \in A$. An elements x of Banach algebra A is said to be invertible if there exists an element y such that $yx = xy = 1$ and is written as $y = x^{-1}$. The set of all invertible elements in an algebra A will be denoted by A^{-1}.

3. PROPERTIES Of THE SPECTRUM

In this section we give spectral characterisation of elements of semisimple Banach algebras.

The spectrum of an element a in A will be denoted by $\sigma(a)$ and is defined by $\sigma(a) = \{\alpha \in \mathbb{C}: a - \alpha \text{ is not invertible}\}$, the spectral radius of a in A by $\mathrm{r}(a)$ and is defined by $\mathrm{r}(a) = \sup\{|\alpha|: \alpha \in \sigma(a)\}$. $\sigma(a)$ is a closed subset of $\mathbb{C}$. The function $\lambda \to (a - \lambda)^{-1}$ defined in the open set $\mathbb{C} \backslash \sigma(a)$ is called the resolvent of a. Also, the resolvent $\lambda \to (a - \lambda)^{-1}$ is analytic in $\mathbb{C} \backslash \sigma(a)$. An element $a \in A$ is said to be quasinilpotent if $\sigma(a) = \{0\}$. The set of quasinilpotent elements will be denoted by $QN(A)$. For more detail and reference on this matter, readers are referred to [1, 2, 7, 8] and to standard textbook, for instance the recent one [5].

If A is a Banach algebra, the set of elements $x \in A$ for which $xz = zx$ for all $z \in A$ is called the centre of A, denoted by $Z(A)$.

Let Δ represents the Hausdorff distance, the spectrum $\sigma(a)$ is said to be Lipschitzian spectrum at a if there exist $0 < R, C \in \mathbb{R}$ such that $\Delta(\sigma(x), \sigma(a)) \leq C\|x - a\|$ for all x satisfying $\|x - a\| < R$.

Theorem 3.1. ([1], Theorem 5.2.1) Let $a \in A$ be such that $\#\sigma(ax - xa) = 1$ for all $x \in A$. Then $a \in Z(A)$.

Theorem 3.2. ([1], Theorem 5.2.2) Let $a \in A$. Then the following properties are equivalent:

(i) $a \in Z(A)$,

(ii) there exists $M > 0$ such that $r(a + x) \leq M(1 + r(x))$, for every $x \in A$,

(iii) there exists $N > 0$ such that $r((a - \lambda 1)^{-1}x) \leq Nr((a - \lambda 1)^{-1})r(x)$ for every $x \in A$ and $\lambda \notin \sigma(a)$.

Theorem 3.2 can be put in a multiplicative form in a semisimple Banach algebra as in the next theorem.

Theorem 3.3. Let A be a semisimple Banach algebra and $a \in A$ such that $r(ax) \leq r(x)$ for all $x \in A^{-1}$. Then $a \in Z(A)$.

Proof. Let $x \in A$ and $\lambda \in \mathbb{C}$ be such that $1 + r(x) < |\lambda|$, then it is clear that $\delta\big(\lambda, \sigma(x)\big) > 1$ and $\lambda \notin \sigma(x)$. Therefore $(\lambda 1 - x)$ is invertible. We also have

$$\lambda 1 - (a + x) = (\lambda 1 - x)(1 - (\lambda 1 - x)^{-1}a) = (\lambda 1 - x)(1 - R_\lambda(x)a).$$

But, by the given condition, we have

$$r(R_\lambda(x)a) \leq r\big(R_\lambda(x)\big) = \frac{1}{\delta\big(\lambda, \sigma(x)\big)} < 1 \text{ ([1], theorem 3.3.5).}$$

Hence $1 - R_\lambda(x)a$ is invertible and therefore $\lambda 1 - (a + x)$ is also invertible. From this discussion we can conclude that $\lambda \notin \sigma(a + x)$. It follows that if $\mu \in \sigma(a + x)$ then $|\mu| \leq 1 + r(x)$. Therefore, $r(a + x) \leq 1 + r(x)$ for all $x \in A$. By theorem 3.2, it is clear that $a \in Z(A)$. ■

Theorem 3.4. ([1], Theorem 5.3.1) (J.Zemanek). Let A be a Banach algebra. Then the following properties are equivalent:

(i) a is in the Jacobson radical of A,

(ii) $\sigma(a + x) = \sigma(x)$, for all $x \in A$,

(iii) $r(a + x) = 0$, for all quasi-nilpotent elements x in A,

(iv) $r(a + x) = 0$, for all quasi-nilpotent elements x in a neighbourhood of 0 in A,

(v) there exists $C > 0$ such that $r(x) \leq C\|x - a\|$, for all x in a neighbourhood of a in A.

Making the use of above theorem we have the following result.

Theorem 3.5. Let A be a semisimple Banach algebra and $a \in A$. If the spectrum is Lipschitzian at a and $\sigma(a) = \{\alpha\}$, then $a = \alpha 1$.

Proof. Since the spectrum is Lipschitzian at a so there exists $0 < R, C \in \mathbb{R}$ such that $\Delta(\sigma(x), \sigma(a)) \leq C\|x - a\|$ for all x satisfying $\|x - a\| < R$. Also Δ is Hausdorff distance, therefore

$$\Delta\big(\sigma(x), \sigma(a)\big) = \Delta(\sigma(x - \alpha 1), \sigma(a - \alpha 1)).$$

Again by given condition $\sigma(a) = \{\alpha\}$, we have $\sigma(a - \alpha 1) = \{0\}$. Taking $x - \alpha 1 = y$ in the distance result, we have

$$\rho(y) = \Delta(\sigma(y), \{0\}) = \Delta\big(\sigma(y), \sigma(a - \alpha 1)\big) \leq C\|x - a\| = C\|y - (a - \alpha 1)\|$$

for all y such that $\|y - (a - \alpha 1)\| < R$. From theorem 3.4., it follows that $a - \alpha 1 \in Rad(A) = \{0\}$. Hence $a = \alpha 1$. ■

Theorem 3.6. Let A be a semisimple Banach algebra and $a \in A$ such that $\sigma(a) = \{\alpha\}$. Then $a = \alpha 1$ if and only if $a \in Z(A)$.

Proof. If $a = \alpha 1$, then it is clear that $a \in Z(A)$.
Conversely, let $a \in Z(A)$. Now $r\big((a - \alpha 1)x\big) \leq r(a - \alpha 1)r(x) = 0$ for all $x \in A$ (since $a - \alpha 1$ and x commute). It follows that $a - \alpha 1 \in Rad(A) = \{0\}$. Thus $a = \alpha 1$. ■

From the above theorem we see that it is sufficient to show that $a \in A$ having single spectrum belongs to $Z(A)$ in order to infer that a is a scalar. In [1] some characterizations of the centre and scalars of a Banach algebra are given, specifically [1], Theorem 5.2.1, 5.2.2, 5.2.4, 5.3.2.

From ([9], Theorem 2), we know that if $a \in A$ satisfies $r(a(1 + q)) = 0$ for all $q \in QN(A)$ then $a \in Rad(A)$.

The following theorem can be seen as an alternative version of ([1], theorem 5.3.2.) which states that an element a in a semisimple Banach algebra A satisfies $\#\sigma(a + q) = 1$ for all $q \in QN(A)$ if and only if a is a scalar.

Theorem 3.7. Let A be a semisimple Banach algebra and $a \in A$ such that $\#\sigma(a+x) \leq \#\sigma(x)$ for all x in a neighbourhood of 1. Then $a = \alpha 1$ $(\alpha \in \mathbb{C})$.

Proof: Let us fix $q \in QN(A)$. From our assumption

$$\#\sigma(a+x) \leq \#\sigma(x) \text{ or } \#\sigma(a+(1+\lambda q)) \leq \#\sigma(1+\lambda q) = 1$$

for all sufficiently small $\lambda \in \mathbb{C}$. From the Scarcity Theorem, it follows that

$$\#\sigma(a+(1+\lambda q)) = 1 \text{ for all } \lambda \in \mathbb{C}.$$

For $\lambda = 1$, we have

$$\#\sigma(a+(1+\lambda q)) = \#\sigma(1+\lambda q) = 1.$$

Since $q \in QN(A)$ was arbitrarily chosen, therefore from ([1], Theorem 5.3.2), the result follows. ■

4. PROPERTIES OF FINITE RANK ELEMENTS

Let A be a semisimple Banach algebra and $a \in A$. Following Aupetit and Mouton [2] we define the rank of a by

$$rank(a) = \sup_{x \in A} \#(\sigma(xa) \backslash \{0\})$$

where # denotes the number of elements in a set. An element $a \in A$ is said to be of maximal finite rank if

$$rank(a) = \#(\sigma(xa) \backslash \{0\}).$$

If $rank(a)$ is finite, then a is said to be a spectrally finite rank element. Therefore if a is spectrally of finite rank, the set

$$E(a) = \{x \in A : \#(\sigma(xa) \backslash \{0\}) = rank(a)\}$$

is non-empty. It is well known that in a semisimple Banach algebra, the set of spectrally finite rank elements coincides with the socle ([2], corollary 2.9).

In [7] we defined an element $0 \neq a \in A$ rank one if $aAa \subset \mathbb{C}a$. We denote the set of these elements by $\mathcal{F}_1$. Also by ([7], Lemma 2.7) we have $A\mathcal{F}_1, \mathcal{F}_1 A \subset \mathcal{F}_1$. An idempotent belonging to $\mathcal{F}_1$ is called a minimal idempotent. Let $\mathcal{F}$ denotes the set of all $u \in A$ of the form $u = \sum_{i=1}^{n} u_i$, where $u_i \in \mathcal{F}_1$. We will call $\mathcal{F}$ the set of finite rank elements of A. $\mathcal{F}$ is a two sided ideal in A and it coincides with the socle of A, i.e., Soc(A) = $\mathcal{F}$.

In the next theorem we prove the result for rank one element. For more detail and different approach to rank one and finite rank elements see [1], [2], [3], [4], [8]. However, if A is a semisimple Banach algebra, then the notion of rank one and finite rank elements in [7] coincides with the notions of [1], [2], [3], [4], [8].

Theorem 4.1. Let A be a semiprime Banach algebra and let $x \in \mathcal{F}_1(A)$. Then x is an idempotent if and only if $r(x) = 1$ and $r(x)$ is a pole of the resolvent of x.

Proof: Since $x \in \mathcal{F}_1(A)$, therefore $x \neq 0$. Thus if x is an idempotent, then $r(x) = 1$. It is well known that the projection $p(e; 1) = e$ for every non zero idempotent e. Therefore $p(x; r(x)) = x$, which implies that $r(x)$ is a simple pole of the resolvent of x.

Conversely, suppose that $r(x) = 1$ and $r(x)$ is a pole of the resolvent of x. Then by Krein-Rutman theorem there exists a $0 \neq u \in A$ such that $xu = ux = u$. Therefore $xux = u$. Since $x \in \mathcal{F}_1(A)$, there exists $0 \neq \lambda\mathbb{C}$ such that $xux = \lambda x$. From $xux = u$ it follows that $u = \lambda x$. Using $ux = u$ we obtain that $\lambda x^2 = \lambda x$, so that $x^2 = x$.

5. CONCLUSIONS

We proved that if A is a semisimple Banach algebra and $a \in A$ has the property that the number of elements in the spectrum of ax is less than or equal to the number of elements in the spectrum of x for all x in an arbitrary neighbourhood of the identity, then a is a scalar. We also proved that an element x is of rank one and idempotent if and only if $r(x) = 1$ and $r(x)$ is a pole of the resolvent of x.

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