MOUFANG ELEMENTS AND THEIR AUTOTOPISM PROPERTIES

Ademola, Lois Adewoye,
School of Mathematics, Universiti Sains Malaysia.

ABSTRACT

In this paper we capture the well defined uniqueness and wholeness of the autotopism property of the Moufang elements.

KEYWORDS:

Loop, Inverse property, Autotopism, Moufang elements.

INTRODUCTION:

An inverse property Loop L, such that its elements satisfy the property $(ux)(yu) = [u(xy)u]$ for all $x, y \in L$ is called a Moufang loop and its elements are called Moufang elements. We shall be considering the autotopism property of the loop L. Several works have been presented by Bruck [1],. Pflugfelder[2], Drapal [3]and others.

We shall however here be considering in details the explicitcomputation of such elements.

BASIC DEFINITIONS:

Definition 1:

A groupoid G is said to have the left inverse property if for each $x \in G$ there is atleast one $a \in G$ such that

$$a(xy) = y \text{ for all } y \in G. \text{ i.e } L(x)L(a) = I$$

G is said to have the right inverse property if for each $x \in G$ there is atleast one $b \in G$ such that $(yx)b = y$ for all $y \in G$. i.e $R(x)R(a) = I$

If G has both the the left inverse property and the the right inverse propert, then G is said to have the inverse property. It is also called IP-loop.

Definition 2:

A triple (α, β, γ) of bijecttions is called an isotopism of Loop $(L,\cdot)$ onto a loop $(H, \circ)$ provided

$$x\ \alpha \circ y\beta = (x \cdot y)\gamma \text{ for all } x, y\ \in L.$$

Definition 3:

An isotopism of $(L,\cdot)$ onto a loop $(L,\cdot)$ is called an autotopism of L and is denoted $A(L)$.

MAIN WORK:

Theorem1:

Let $\mathrm{A(p)} = (\mathrm{L(p)}, \mathrm{R(p)}, \mathrm{L(p)R(p)})$

Claim 1:

A(p) is an autotopism and so is $A(p)^{-1}$.

Proof of Claim 1:

By definition *** we have that :

$$xL(p) \cdot yR(p) = (px) \cdot (yp) = [p(xy)p] = (xy)L(p)R(p).$$

Therefore A(p) is an autotopism

Claim 2:

$A(p)^{-1}$ is an autotopism

Proof of Claim 2:

Now

$$A(p)^{-1} = [\mathrm{L(p)}, \mathrm{R(p)}, \mathrm{L(p)R(p)}]^{-1}$$

$$= (L(p)^{-1}, R(p)^{-1}, [\mathrm{L(p)R(p)}]^{-1})$$

Claim 2i:

$L(p)^{-1} = L(p^{-1})$ and $R(p)^{-1} = R(p^{-1})$

Proof of Claim 2i:

Now the fact $(xy)^{-1} = y^{-1}x^{-1}$ obviously means $R(x)^{-1} = R(x^{-1})$

In the same argument $(yx)^{-1} = x^{-1}y^{-1}$ means $L(x)^{-1} = L(x^{-1})$

So we have by definition *** that and the proof of Claim2i that:

$$xL(p)^{-1} \cdot yR(p)^{-1} = xL(p^{-1}) \cdot yR(p^{-1})$$

$$= (p^{-1}x) \cdot (yp^{-1})$$

$$= p^{-1}(xy)p^{-1}$$

$$= (xy)L(p^{-1})R(p^{-1})$$

$$= (xy)L(p)^{-1}R(p)^{-1}$$

Therefore $A(p)^{-1}$ is an autotopism.

Hence the prove of theorem 1.

Theorem 2:

Let A(p) be as define in Theorem 1;

Claim 2:

i. $\mathrm{A(q,p)}$ is an autotopism
ii. $A(p)^{-1}A(q)^{-1}A(q^{-1}p^{-1})^{-1}$ is an autotopism

Proof of Claim 2i:

A similar step with the proof for Theorem 1 will be used here but we would be requiring the knowledge of component wise multiplication.

So we would have that:

$$\mathrm{A(q,p) = A(p)A(q)}$$

$$= [\mathrm{L(p)}, \mathrm{R(p)}, \mathrm{L(p)R(p)}][\mathrm{L(q)}, \mathrm{R(q)}, \mathrm{L(q)R(q)}]$$

$$= [\mathrm{L(q)L(p)}, \mathrm{R(q)R(p)}, \mathrm{L(q)R(q)L(p)R(p)}]$$

$$= \mathrm{xL(q)L(p)} \cdot y\mathrm{R(q)R(p)}$$

$$= pqx \cdot yqp$$

$$= pq(xy)qp$$

$$= (xy)\mathrm{L(q)R(q)L(p)R(p)}$$

Thus $\mathrm{A(q, p)}$ is an autotopism.

Proof of Claim 2ii:

Now

$$A(p)^{-1} = [\mathrm{L(p)}, \mathrm{R(p)}, \mathrm{L(p)R(p)}]^{-1}$$

$$= [L(p^{-1}), R(p^{-1}), L(p^{-1})R(p^{-1})]$$

$$A(q)^{-1} = [\mathrm{L(q)}, \mathrm{R(q)}, \mathrm{L(q)R(q)}]^{-1}$$

$$= [L(q^{-1}), R(q^{-1}), L(q^{-1})R(q^{-1})]$$

$$A(q^{-1}p^{-1})^{-1} = [\mathrm{L}(q^{-1}p^{-1}), \mathrm{R}(q^{-1}p^{-1}), \mathrm{L}(q^{-1}p^{-1})\mathrm{R}(q^{-1}p^{-1})]^{-1}$$

$$= [L(q^{-1}p^{-1})^{-1}, R(q^{-1}p^{-1})^{-1}, L(q^{-1}p^{-1})^{-1}R(q^{-1}p^{-1})^{-1}$$

Our goal is to show that $A(p)^{-1}A(q)^{-1}A(q^{-1}p^{-1})^{-1}$ is an autotopism,

So we have that ;

$A(p)^{-1}A(q)^{-1}A(q^{-1}p^{-1})^{-1}$
$= [\mathrm{L(p)}, \mathrm{R(p)}, \mathrm{L(p)R(p)}]^{-1}[\mathrm{L(q)}, \mathrm{R(q)}, \mathrm{L(q)R(q)}]^{-1}[L(q^{-1}p^{-1})^{-1}, R(q^{-1}p^{-1})^{-1}, L(q^{-1}p^{-1})^{-1}R(q^{-1}p^{-1})^{-1}]$

$= [L(p^{-1}), R(p^{-1}), L(p^{-1})R(p^{-1})][L(q^{-1}), R(q^{-1}), L(q^{-1})R(q^{-1})][L(q^{-1}p^{-1})^{-1}, R(q^{-1}p^{-1})^{-1}, L(q^{-1}p^{-1})^{-1}R(q^{-1}p^{-1})^{-1}$

$= [L(p^{-1})L(q^{-1})\, L(q^{-1}p^{-1})^{-1}, R(p^{-1})R(q^{-1})\, R(q^{-1}p^{-1})^{-1}, L(p^{-1})R(p^{-1})L(q^{-1})R(q^{-1})L(q^{-1}p^{-1})^{-1}R(q^{-1}p^{-1})^{-1}]$

Set

$\mathrm{S} = L(p^{-1})L(q^{-1})\, L(q^{-1}p^{-1})^{-1} = L(p^{-1}, q^{-1}),$

$T = R(p^{-1})R(q^{-1})\, R(q^{-1}p^{-1})^{-1} = R(p^{-1}, q^{-1}),$

$X = L(p^{-1})R(p^{-1})L(q^{-1})R(q^{-1})L(q^{-1}p^{-1})^{-1}R(q^{-1}p^{-1})^{-1} = ST.$

So using definition *** we have that:

$$\mathrm{xS} \cdot yT = (q^{-1}p^{-1})^{-1}q^{-1}p^{-1}x \cdot yp^{-1}q^{-1}(q^{-1}p^{-1})^{-1}$$

$$= [(q^{-1}p^{-1})^{-1}q^{-1}p^{-1}](xy)[p^{-1}q^{-1}(q^{-1}p^{-1})^{-1}]$$

$$= (xy)\, L(p^{-1})R(p^{-1})L(q^{-1})R(q^{-1})L(q^{-1}p^{-1})^{-1}R(q^{-1}p^{-1})^{-1}$$

$$= (xy)X$$

Thus $A(p)^{-1}A(q)^{-1}A(q^{-1}p^{-1})^{-1}$ is an autotopism as required.

CONCLUSION:

The aspect of autotopism of Loops is today a very viable research area as it extends to Pseudo-automorphisms. This paper has succeeded in presenting in details and emphasising major aspects in autotopism.

REFERENCES:

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2. H.O. Pflugfelder (1990), Quasigroups and loops
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